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B-type defects in Landau-Ginzburg models

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Abstract: We consider Landau-Ginzburg models with possibly different superpotentials glued together along one-dimensional defect lines. Defects preserving B-type supersymmetry can be represented by matrix factorisations of the difference of the superpotentials. The composition of these defects and their action on B-type boundary conditions is described in this framework. The cases of Landau-Ginzburg models with superpotential $W = X^d$ and $W = X^d + Z^2$ are analysed in detail, and the results are compared to the CFT treatment of defects in $N = 2$ superconformal minimal models to which these Landau-Ginzburg models flow in the IR.

Keywords: D-branes, Topological Field Theories.

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Contents

1. Introduction

Over the last decades, D-branes and their descriptions from various points of views such as boundary states in conformal field theory or solutions to supergravity equations have received a lot of attention. An interesting and quite useful property of D-branes is that there exist operations which act naturally on them and relate D-branes in possibly different theories. Examples of such operations are dualities such as T-duality or mirror symmetry, but also monodromy transformations along paths in moduli spaces. Some of those operations, namely the natural operations on the category of B-type D-branes in Calabi-Yau compactifications, have an elegant description in terms of Fourier-Mukai transformations. All these examples have in common that they do not involve the string coupling g_s and hence can be studied at weak coupling in the framework of conformal field theory.

From a world sheet point of view, a natural operation on D-branes is provided by defects, one-dimensional interfaces, along which two possibly different conformal field theories are glued together. Defects in conformal field theory have received some attention recently [1 – 9] and have also emerged as domain walls in the discussion of the $\mathrm{AdS}_3/\mathrm{CFT}_2$-duality [2].

Among the defects in conformal field theories, there is a special class of so-called *topological* defects, which have the property that they can be moved freely across the world sheet, as long as they do not cross field insertions or other defects. They act naturally on conformal boundary conditions, i.e. D-branes. Namely, when both defects and boundaries are present one can bring defects close to the world sheet boundary and in the limit in which the defect approaches the boundary, a new boundary condition arises. To put it differently, the world sheet boundary couples to a new D-brane. Likewise, one can bring together two topological defects. In the limit in which the two merge, one obtains new defects, and hence topological defects can be composed [1, 6].

Generic defects however cannot be composed or act on boundary conditions in this way. In general, correlation functions in the presence of defects depend on the positions of the latter, and in particular exhibit singularities when defects approach each other or world sheet boundaries. So the process of merging defects, or moving defects to world sheet boundaries is a priori not well defined for generic conformal defects.

Of course, defects can also be studied in the context of topological rather than conformal field theory. In topological field theories, where correlation functions do not depend on the world sheet metric, defects can always be moved and hence can always merge and act on the D-branes of the topological theory. Therefore, *any* defect in topological field theory provides a suitable map between the respective D-brane categories.

This is in particular true for defects in topologically twisted $N = (2, 2)$ supersymmetric field theories (including superconformal field theories). As will be described in section 2, supersymmetry preserving defects in $N = (2, 2)$ supersymmetric field theories come in two variants, just as D-branes or orientifolds do. A-type defects are compatible with the topological A-twist, B-type defects with the topological B-twist. Hence the topological twist endows both of these types of defects with a composition and action on the respective class of D-branes, even though they are not topological in the untwisted field theory. Unlike D-branes or orientifolds, defects can be of A- and B-type at the same time. Those defects are topological already in the untwisted theory.

Our main focus in this paper will be the investigation of B-type defects in $N = (2, 2)$ supersymmetric Landau-Ginzburg models. These flow to superconformal field theories in the IR, and play an important role in the study of string compactifications on Calabi-Yau manifolds, where they provide useful descriptions of the small volume regime.

We consider the situation where a Landau-Ginzburg model with superpotential W_1 is

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separated by a defect from a Landau-Ginzburg model with superpotential W_2. We argue that similar to B-branes in these models, which can be represented by matrix factorisations of the W_i [10−12], B-type defects between the models are described by matrix factorisations of the difference $W_1 - W_2$ of the superpotentials (see [13, 14] for earlier work that has discussed these defects from a slightly different point of view). We then give a prescription of the composition of these defects, and their action on the respective boundary conditions in this framework.

We will discuss in particular a simple class of defects, which do not introduce any additional degrees of freedom. Such defects are related to symmetries and indeed orbifolds of the underlying bulk theories. They implement the action of these symmetries on bulk fields, and their defect-changing operators correspond to twisted sectors in the respective orbifold models.

For the simple classes of Landau-Ginzburg models with a single chiral superfield and superpotential $W = X^d$ and their cousins[1] with an additional superfield and superpotential $W = X^d + Z^2$, we compare the description of B-type defects, their composition and action on B-type boundary conditions in the framework of matrix factorisations with the respective conformal field theory description available in the IR. Namely, these Landau-Ginzburg models flow to $N = 2$ superconformal minimal models in the IR, in which defects can be studied by means of CFT techniques. We find complete agreement between the two approaches.

This paper is organised as follows. In section 2 we set the stage and discuss general properties of supersymmetry preserving defects in theories with $N = (2, 2)$ supersymmetry. section 3 is devoted to the study of defects in Landau-Ginzburg models. In particular, we show that supersymmetry preserving B-type defects in Landau-Ginzburg models are described by matrix factorisations. As a next step, in section 4 we consider situations where several defects or both defects and boundaries are present, and work out the composition of defects and their action on boundary conditions in this framework. The special class of symmetry defects is discussed in section 5. section 6 contains the explicit comparison of B-type defects in Landau-Ginzburg with superpotentials $W = X^d$, $W = X^d + Z^2$ and defects in the corresponding superconformal minimal models. Some technical details appear in the appendix.

2. Defects in $N = 2$ theories

In this paper we consider two-dimensional field theories with $N = 2$ supersymmetry for both left and right moving degrees of freedom. There are hence four anti-commuting supercharges $Q_\pm$, $\bar{Q}_\pm$ satisfying the usual anti-commutation relations

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P \,, \tag{2.1}$$

with all other anti-commutators vanishing. H and P denote energy and momentum density, the superscripts $^\pm$ distinguish left and right movers and a bar indicates conjugation.

[1] Although the bulk Landau-Ginzburg theory hardly changes when adding a square to the superpotential, the B-brane spectra of the two theories are rather different [15].

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We are interested in situations where two such theories are glued together along a common one-dimensional interface, a *defect*. Our focus will be on supersymmetry preserving defects, i.e. those defects whose presence still allows the total theory to be supersymmetric with respect to one half of the supersymmetries present in the original bulk theories. Just like in the case of $N = 2$ theories on surfaces with boundaries or unoriented surfaces, there are two ways of doing so. The respective defects are called *A*- and *B-type* respectively. Modelling the defect on the real line $\mathbb{R} \subset \mathbb{C}$ separating two possibly different theories on the upper and lower half plane, *B-type defects* have the property that the combination $Q_B = Q_+ + Q_-$ of supercharges and its conjugate $\bar{Q}_B$ are preserved everywhere on $\mathbb{C}$. That means that along the interface $\mathbb{R}$ the supercharges have to satisfy the following "gluing conditions":

$$\begin{aligned} Q_+^{(1)} + Q_-^{(1)} &= Q_+^{(2)} + Q_-^{(2)}, \\ \bar{Q}_+^{(1)} + \bar{Q}_-^{(1)} &= \bar{Q}_+^{(2)} + \bar{Q}_-^{(2)} \ . \end{aligned} \tag{2.2}$$

Here, the superscripts $^{(1)}$ and $^{(2)}$ refer to the two theories on upper and lower half plane respectively. For *A-type defects* on the other hand, the gluing conditions along the defect are twisted by the automorphism of the supersymmetry algebra which exchanges $Q_\pm$ with $\bar{Q}_\pm$:

$$\begin{aligned} Q_+^{(1)} + \bar{Q}_-^{(1)} &= Q_+^{(2)} + \bar{Q}_-^{(2)}, \\ \bar{Q}_+^{(1)} + Q_-^{(1)} &= \bar{Q}_+^{(2)} + Q_-^{(2)}. \end{aligned} \tag{2.3}$$

They ensure that the combination $Q_A = Q_+ + \bar{Q}_-$ and its conjugate $\bar{Q}_A$ are preserved.[2]

In situations where defects as well as boundaries are present, A- or B-type supersymmetry can be preserved in case all defects and all boundaries are of A- or B-type respectively. Just as for D-branes, mirror symmetry exchanges A- and B-type defects.

Note that there are two special classes of defects which actually preserve the full $N = (2, 2)$ algebra. The first class consists of defects such that

$$Q_\pm^{(1)} = Q_\pm^{(2)}, \quad \bar{Q}_\pm^{(1)} = \bar{Q}_\pm^{(2)} \quad \text{on } \mathbb{R}\,, \tag{2.4}$$

which implies both A- as well as B-type gluing conditions (2.3), (2.2). One particular defect of this kind is the trivial defect between one and the same theory. Defects of the second class are related to those of the first class by mirror symmetry. They obey the respective mirror twisted gluing conditions

$$\begin{aligned} Q_+^{(1)} &= Q_+^{(2)}, \quad \bar{Q}_+^{(1)} = \bar{Q}_+^{(2)} \\ Q_-^{(1)} &= \bar{Q}_-^{(2)}, \quad \bar{Q}_-^{(1)} = Q_-^{(2)} \quad \text{on } \mathbb{R}\ . \end{aligned} \tag{2.5}$$

[2] Of course, there are also other automorphisms of the $N = (2, 2)$ supersymmetry algebra, which can be used to twist the gluing conditions. For $\eta^\pm \in \{\pm 1\}$, $Q_\pm \mapsto \eta^\pm Q_\pm$, $\bar{Q}_\pm \mapsto \eta^\pm \bar{Q}_\pm$ gives rise to modified A- and B-type gluing conditions. For simplicity of presentation we will refrain from spelling out the details of these additional possibilities here, but we will comment on $\eta^\pm$-twisted gluing conditions in the context of conformal field theory in section 6.2.

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Such defects exist for example between a theory and its mirror and hence realise mirror symmetry as a defect.

Using the supersymmetry algebra, it follows immediately that defects of these two classes preserve translational invariance in space and time because the gluing conditions for the supercharges imply

$$H^{(1)} = H^{(2)}, \quad P^{(1)} = P^{(2)} \quad \text{on } \mathbb{R} \,. \tag{2.6}$$

This is not possible for world sheet boundaries which automatically break one half of the local translation symmetries and therefore can at most preserve half of the bulk supersymmetries. In contrast, a theory with a defect allows for the possibility of being invariant under shifts of the defect on the world sheet.

Nevertheless, the similarities between defects and boundaries are indeed very useful for the treatment of defects. In particular, one can obtain an equivalent description of the situation described above by folding the world sheet along the real line and realising the degrees of freedom of the theories on the upper and lower half plane as different sectors in a "doubled" theory defined on the upper half plane only [2, 16]. Folding the theory from the lower to the upper half plane, left and right movers are interchanged, and defects in the original theory on the complex plane become boundary conditions in the doubled theory. If the defect preserves the full $N = (2,2)$ supersymmetry, the corresponding boundary conditions in the doubled theory are of permutation type, i.e. left movers of the supercharges in one sector are glued to right movers of the respective supercharges in the other one and vice versa.

Of particular interest in the context of string theory are theories with $N = (2,2)$ superconformal symmetry. The corresponding symmetry algebra is generated by the modes of the energy momentum tensor T, U(1)-current J and two supercurrents $G^{\pm}$ together with the ones of the respective right movers $\overline{T}$, $\overline{J}$, $\overline{G}^{\pm}$. (As is customary in CFT, the superscripts $^{\pm}$ specify the U(1)-charge of the respective current, and right movers will be distinguished from left movers by a bar. This differs from the notation used for the supercharges Q in the discussion above.)

In these theories, one can consider defects preserving one half of the bulk superconformal symmetry. As before, we call them A- and B-type depending on which combinations of supercharges are conserved. The corresponding gluing conditions along the real line are given by

$$\begin{aligned} T^{(1)} - \overline{T}^{(1)} &= T^{(2)} - \overline{T}^{(2)} \,, \\ J^{(1)} - \overline{J}^{(1)} &= J^{(2)} - \overline{J}^{(2)} \,, \\ G^{\pm(1)} + \overline{G}^{\pm(1)} &= G^{\pm(2)} + \overline{G}^{\pm(2)} \end{aligned} \tag{2.7}$$

for B-type defects and

$$\begin{aligned} T^{(1)} - \overline{T}^{(1)} &= T^{(2)} - \overline{T}^{(2)} \,, \\ J^{(1)} + \overline{J}^{(1)} &= J^{(2)} + \overline{J}^{(2)} \,, \\ G^{\pm(1)} + \overline{G}^{\mp(1)} &= G^{\pm(2)} + \overline{G}^{\mp(2)} \end{aligned} \tag{2.8}$$

for A-type defects.

Just as in the general situation, there is also a class of defects that preserves the full $N = (2, 2)$ superconformal symmetry, and is hence both of A- as well as B-type. Because of the two automorphisms of the $N = 2$ superconformal algebra there are essentially four possible gluing conditions for those defects. Namely, for $a, \bar{a} \in \{\pm 1\}$ we have

$$\begin{aligned} T^{(1)} - T^{(2)} &= 0 = \overline{T}^{(1)} - \overline{T}^{(2)}\,, \\ J^{(1)} - aJ^{(2)} &= 0 = \overline{J}^{(1)} - \bar{a}\overline{J}^{(2)}\,, \\ G^{\pm(1)} + G^{a\pm(2)} &= 0 = \overline{G}^{\pm(1)} + \overline{G}^{\bar{a}\pm(2)}\,. \end{aligned} \tag{2.9}$$

These defects in particular glue together holomorphic and antiholomorphic energy momentum tensors separately and therefore preserve both holomorphic and antiholomorphic Virasoro algebras. This implies that despite the presence of such a defect, correlation functions are still covariant with respect to all local conformal transformations of the world sheet, even those which change the position of the defect. This implies that correlation functions do not change when such defects are shifted on the world sheet. Defects which have this property have been called *topological* in [5].

Since they can be shifted on the world sheet, topological defects can in particular be brought on top of each other, to "fuse" to new defects. This procedure furnishes the topological defects with a composition. Moreover, they can also be brought on top of world sheet boundaries producing new boundary conditions in this way. Hence, topological defects act on boundary conditions. This is not true for non-topological defects. Letting two of those defects approach each other, or one of them approach a boundary will in general lead to singularities in correlation functions.

Certain $N = (2, 2)$ supersymmetric field theories, in particular those considered here can be topologically twisted [17]. Twisting changes the energy momentum tensor of the theory in such a way that it is exact with respect to a BRST-operator $Q = Q_A$ for an A-twisted theory or $Q = Q_B$ for a B-twisted theory. A consequence of this is that correlation functions only involving Q-closed fields are invariant with respect to variations of the world sheet metric and thus define a topological field theory. This twisting procedure is compatible with the existence of boundaries and defects, as long as the chosen BRST-charge is preserved by the boundaries and defects. More precisely, A- and B-twisting is compatible with A- and B-type boundary conditions and defects respectively. Indeed, by arguments similar to those used for pure bulk theories it follows that also correlation function of BRST-closed fields in the presence of boundaries and defects become topological in the twisted theory. In particular, upon twisting all defects, even those which have not been topological in the original untwisted theory become topological, i.e. in the topologically twisted theory they can be shifted on the world sheet. Thus, the topological twisting provides a composition of all A- and B-type defects and an action of them on A- and B-type boundary conditions respectively. We will study the action of B-type defects in Landau-Ginzburg models and their action on B-type boundary conditions below.

Let us close this section with a few general remarks about defects. Similarly to boundary conditions, defects add to the structure of the underlying bulk theories. For instance, if a defect is located on a closed curve, such that the world sheet can be cut open on both sides of it, the defect provides a homomorphism between the bulk Hilbert spaces of the theories it separates. (This is similar to boundary conditions giving rise to boundary states.) These defect operators are often a convenient way to encode part of the information about a defect, and we will make use of it below. Note however that there is more structure. Similar to boundary conditions which come with additional degrees of freedom such as boundary condition changing boundary fields (open strings between the respective D-branes), also defects introduce new degrees of freedom. Unlike boundaries however, defects can form junctions, and there are fields localised on all possible junctions of defects (including the one-junction, which is a tip of a defect). In a more string theoretic language one would call these degrees of freedom closed strings twisted by the respective defects. If moreover there are boundaries and defects in a theory, defects can also end on boundaries, giving rise to even more degrees of freedom etc. Part of these structures will be described in explicit examples below.

3. Defects in Landau-Ginzburg models

3.1 Bulk action

Our conventions for the $N = (2, 2)$ superspace are those of [18]. The two-dimensional $(2, 2)$ superspace is spanned by two bosonic coordinates $x^\pm = x^0 \pm x^1$ and four fermionic coordinates $\theta^\pm$, $\bar\theta^\pm$. The supercharges are realised as the following differential operators on superspace

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\bar\theta^\pm\partial_\pm\,, \qquad \bar Q_\pm = -\frac{\partial}{\partial\bar\theta^\pm} - i\theta^\pm\partial_\pm\,.$$

The superderivatives are given by

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm\,, \qquad \bar D_\pm = -\frac{\partial}{\partial\bar\theta^\pm} + i\theta^\pm\partial_\pm\,.$$

Chiral superfields X satisfy the conditions $\bar D_\pm X = 0$ and have an expansion

$$X = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm) \tag{3.1}$$

into components, where $y^\pm = x^\pm - i\theta^\pm\bar\theta^\pm$ and $\alpha \in \{\pm\}$. The conjugate fields $\bar X$ are anti-chiral, i.e. they satisfy $D_\pm\bar X = 0$.

We consider Landau-Ginzburg models with a finite number of chiral superfields X_i and action given by the sum

$$S = S_D + S_F \tag{3.2}$$

of D- and F-term. The D-term

$$S_D = \int d^4\theta d^2x K(X_i, \bar X_i) \tag{3.3}$$

is determined by the Kähler potential K which we will assume to be flat and diagonal, $K = \sum_i \bar X_i X_i$. In the topologically twisted theory, the variation of a D-term is BRST trivial and therefore all correlation functions are independent of D-term changes. This is well-known for the case of world sheets without boundary, and has been extended to world sheets with boundary in [19] . The F-term

$$S_F = \int d^2x d\theta^- d\theta^+ W(X_i)|_{\bar{\theta}^\pm=0} \; + \int d^2x d\bar{\theta}^+ d\bar{\theta}^- \overline{W}(\bar{X}_i)|_{\theta^\pm=0} \tag{3.4}$$

is parametrised by the superpotential W, a holomorphic function of the chiral superfields X_i. It is this term, which completely determines the B-twisted model, and we will therefore focus our discussions on it. In the case that W is quasi-homogeneous the Landau-Ginzburg model will flow to a conformal field theory in the IR. According to standard arguments, the D-term will adjust itself in this process to be compatible with the conformal symmetry, whereas the F-term remains unrenormalised. Therefore, for comparisons with conformal field theory only the F-term will be relevant.

On a world sheet without boundaries or defects, the Landau-Ginzburg action is manifestly $N = (2, 2)$ supersymmetric, i.e. the variation of the action with respect to

$$\delta = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+ \tag{3.5}$$

vanishes for all $\epsilon_\pm$, $\bar{\epsilon}_\pm$. The corresponding conserved supercharges can be realised as

$$\begin{aligned} Q_\pm &= \int dx^1 \big((\partial_0 \pm \partial_1) \bar{\phi}_{\bar{j}} \psi^j_\pm \mp i \bar{\psi}^{\bar{i}}_\mp \partial_{\bar{i}} \overline{W} \big) \, , \\ \bar{Q}_\pm &= \int dx^1 \big(\bar{\psi}^{\bar{j}}_\pm (\partial_0 \pm \partial_1) \phi_j \pm i \psi^i_\mp \partial_i W \big) \, . \end{aligned} \tag{3.6}$$

3.2 B-type boundary conditions and matrix factorisations

Let us briefly review the formulation of a Landau-Ginzburg theory on the upper half plane (UHP) [20, 10, 21, 11, 12]. We will consider the situation where superspace acquires a B-type superboundary with coordinates

$$x^+ = x^- = t, \quad \theta^+ = \theta^- = \theta, \quad \bar{\theta}^+ = \bar{\theta}^- = \bar{\theta} \, . \tag{3.7}$$

The presence of the boundary reduces the number of supersymmetries of the theory, because only the combinations

$$\delta_B = \epsilon Q - \bar{\epsilon} \bar{Q}, \tag{3.8}$$

of the supersymmetry generators with

$$Q = Q_+ + Q_-, \qquad \bar{Q} = \bar{Q}_+ + \bar{Q}_- \tag{3.9}$$

are compatible with the B-type boundary. To put it differently, a supersymmetry of the form (3.5) only preserves the boundary if $\epsilon_+ = -\epsilon_- =: \epsilon$ and $\bar{\epsilon}_+ = -\bar{\epsilon}_- =: \bar{\epsilon}$.

As it turns out, the restriction of the bulk Landau-Ginzburg action to a world sheet with B-type boundary on its own is not invariant under the B-type supersymmetry (3.8). Namely, the δ_B-variation of the bulk Landau-Ginzburg action (3.2) in the presence of the boundary introduces boundary terms

$$\delta_B S = \delta_B S_D + \delta_B S_F \, , \tag{3.10}$$

where in particular the variation of the F-term yields [20, 18]

$$\delta_B S_F = i\int_{\partial\Sigma} dt d\theta \bar{\epsilon} W - i\int_{\partial\Sigma} dt d\bar{\theta}\epsilon\bar{W}\,. \tag{3.11}$$

Thus, in order to define a supersymmetric theory on a surface with boundary, one either has to impose boundary conditions on the fields, which ensure the vanishing of (3.10), or add an additional boundary term to the action whose supersymmetry variation compensates for the term coming from the bulk variation. In fact, it has been argued in [11, 12] that the D-term in (3.10) can always be compensated by the supersymmetry variation of an appropriately chosen boundary term, and that the F-term (3.11) can be cancelled by introducing extra non-chiral fermionic boundary superfields $\pi_1, \ldots, \pi_r$ satisfying

$$\bar{D}\pi_i = E_i\ . \tag{3.12}$$

Indeed, the supersymmetry variation of the boundary F-term

$$\Delta S = i\int_{\partial\Sigma} dt d\theta J_i \pi_i \ + \ c.c \tag{3.13}$$

exactly cancels the term (3.11) resulting from the supersymmetry variation of the bulk F-term if

$$\sum_i J_i E_i = W\,. \tag{3.14}$$

Therefore, any factorisation (3.14) of the superpotential W gives rise to a supersymmetric action of the Landau-Ginzburg model on a surface with B-type boundary. To put it differently, such a factorisation defines a supersymmetric B-type boundary condition of the model. We will omit the discussion of the kinetic terms for the boundary fermions, since they play no role in the current context.

Physically, the D-branes constructed in this way are composites of a brane-anti-brane pair obtained by a tachyon condensation. To be more precise, the brane-anti-brane pair is a pair of flat space-filling D-branes in the theory with vanishing superpotential $W = 0$ — a sigma-model with target space $\mathbb{C}^N$, where N is the number of chiral superfields. The tachyon condensation is triggered by turning on the superpotential W. In this picture, fermionic degrees of freedom correspond to strings stretching from brane to anti-brane. In particular, the fermionic matrix Q contains the tachyon profile on the space-time filling brane-anti-brane pair.

As in the case of Landau-Ginzburg theories on surfaces without boundary, one can also perform a topological twist in the presence of supersymmetric boundaries to extract information about the topological sectors. However, only the topological B-twist is compatible with B-type boundary conditions. The BRST-charge of the corresponding B-model in the presence of the B-type boundary condition defined by (3.14) then receives a boundary contribution

$$Q_{\rm bd} = \sum_i J_i \pi_i + E_i \bar{\pi}_i\,, \tag{3.15}$$

which obeys $Q_{\rm bd}^2 = W$ by means of the factorisation condition (3.14). As usual, the degrees of freedom of the twisted theory are given by the cohomology of the BRST-operator. In particular, the topological boundary fields are described by the cohomology of the boundary BRST-operator, which acts on boundary fields by the graded commutator with $Q_{\rm bd}$. (The $\mathbb{Z}_2$-grading is due to the presence of bosonic and fermionic degrees of freedom on the boundary.)

Two such B-type boundary conditions defined by boundary BRST-charges $Q_{\rm bd}$ and $Q'_{\rm bd}$, are equivalent, if there are homomorphisms U and V between the respective spaces of boundary fields, which preserve the $\mathbb{Z}_2$-grading such that

$$Q'_{\rm bd} = U Q_{\rm bd} V\,, \qquad UV = {\rm id}' + \{Q'_{\rm bd}, O'\}\,, \quad VU = {\rm id} + \{Q_{\rm bd}, O\} \tag{3.16}$$

for some O and O'. U corresponds to an open string operator propagating from one brane to the other which can be composed with the "inverse" V propagating in the other direction to yield the identity operators on both of the individual branes.

Note that the notion of equivalence in the B-brane category only requires U and V to be inverse up to BRST-trivial terms.[3] One consequence of this is that all physically trivial matrix factorisations, i.e. those associated to D-branes which do not have any non-trivial open strings ending on them, are mutually equivalent. One particular representative of this trivial factorisation can be obtained by setting $r = 1$, $J_1 = 1$ and $E_1 = W$. In the language of the covering theory with $W = 0$, this amounts to a trivial brane-anti-brane pair. Adding it to any other boundary condition does not change the physical content, and hence gives rise to an equivalent boundary condition:

$$Q_{\rm bd} \sim Q_{\rm bd} \oplus Q_{\rm triv}\,. \tag{3.17}$$

Choosing an explicit matrix representation of the Clifford algebra generated by the boundary fermions π_i, the boundary BRST-charges $Q_{\rm bd}$ are represented by $2^{r+1} \times 2^{r+1}$-matrices of the form

$$Q_{\rm bd} = \begin{pmatrix} 0 & p_1 \\ p_0 & 0 \end{pmatrix}\,, \tag{3.18}$$

where the p_i are $2^r \times 2^r$-matrices whose entries are polynomials in the chiral fields X_i such that $p_1 p_0 = W(X_i){\rm id}_{2^r\times 2^r} = p_0 p_1$. The p_i constitute a *matrix factorisation* of W of rank 2^r and determine $Q_{\rm bd}$ and hence the B-type boundary condition. More generally also $Q_{\rm bd}$ constructed out of matrix factorisations p_1, p_0 of arbitrary rank N define meaningful boundary conditions. This has been shown in [19] by taking into account the gauge degrees of freedom in higher multiplicity brane configurations in the underlying $\mathbb{C}^N$-sigma model.

One often represents matrix factorisations in the following way [21]

$$P: \quad P_1 = \mathbb{C}[X_i]^N \underset{p_0}{\overset{p_1}{\rightleftarrows}} \mathbb{C}[X_i]^N = P_0\,, \qquad p_1 p_0 = W(X_i){\rm id}_{P_0}\,, \quad p_0 p_1 = W(X_i){\rm id}_{P_1}\,. \tag{3.19}$$

[3] It is also possible to consider related categories, in which the notion of equivalence is different from the one used here. For instance instead of taking as morphism spaces the BRST-cohomology, one could use the space of BRST-closed operators. In order to define an equivalence in this category U and V would have to be genuine inverses of each other.

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As follows from (3.16), two such matrix factorisations P and P' lead to equivalent boundary conditions, if there exist homomorphisms $u_i : P_i \to P'_i$, $v_i : P'_i \to P_i$ such that

$$p'_1 = u_0 p_1 v_1 \,, \quad p'_0 = u_1 p_0 v_0 \,, \quad p_1 = v_0 p'_1 u_1 \,, \quad p_0 = v_1 p'_0 u_0 \tag{3.20}$$

and

$$\begin{aligned} v_0 u_0 &= \mathrm{id}_{P_0} + \chi_1 p_0 + p_1 \chi_0 \,, \quad v_1 u_1 = \mathrm{id}_{P_1} + p_0 \chi_1 + \chi_0 p_1 \,, \\ u_0 v_0 &= \mathrm{id}_{P'_0} + \chi'_1 p'_0 + p'_1 \chi'_0 \,, \quad u_1 v_1 = \mathrm{id}_{P'_1} + p'_0 \chi'_1 + \chi'_0 p'_1 \,, \end{aligned} \tag{3.21}$$

for some $\chi_i : P_i \to P_{i+1}$, $\chi'_i : P'_i \to P'_{i+1}$.

In this language, the class of trivial boundary conditions mentioned above can be represented by the rank-one matrix factorisations

$$T : \quad P_1 = \mathbb{C}[X_i] \underset{p_0 = W}{\overset{p_1 = 1}{\rightleftarrows}} \mathbb{C}[X_i] = P_0 \,. \tag{3.22}$$

Any trivial matrix factorisation is equivalent to this special representative.

As mentioned above, the topological boundary degrees of freedom are described by the cohomology of the boundary BRST-operator. In terms of matrix factorisations the boundary BRST-operator on the boundary condition changing sector between boundary conditions defined by matrix factorisations P and P' (topological open strings between the D-branes associated to P and P') is given by the graded commutator with Q_{bd} on the space $\mathrm{Hom}_{\mathbb{C}[X_i]}(P_1 \oplus P_0, {P_1}' \oplus {P_0}')$ of boundary changing fields. More precisely, this operator acts on a boundary condition changing field $\varphi \in \mathrm{Hom}_{\mathbb{C}[X_i]}(P_1 \oplus P_0, {P_1}' \oplus {P_0}')$ by

$$\varphi \mapsto Q'_{\mathrm{bd}} \varphi - \sigma' \varphi \sigma Q_{\mathrm{bd}} \,, \tag{3.23}$$

where $\sigma = \mathrm{id}_{P_0} - \mathrm{id}_{P_1}$ is the grading operator on P. Since the BRST operator respects the grading, also its cohomology $\mathcal{H}(P, P') = \mathcal{H}^0(P, P') \oplus \mathcal{H}^1(P, P')$ is graded.

In the following, we will mostly be interested in the case that W is quasi-homogeneous, i.e. $W(\lambda^{q_i} X_i) = \lambda^q W(X_i)$ for some weights q_i, q, because these superpotentials directly correspond to the superconformal field theories in the IR.[4] For such W, one can also consider quasi-homogeneous matrix factorisations, i.e. matrix factorisations

$$P : \quad P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \,. \tag{3.24}$$

together with representations ρ_i of $\mathbb{C}^*$ on the modules P_i which are compatible with the $\mathbb{C}[X_i]$-action, such that the maps p_i are quasi-homogeneous:

$$\rho_0(\lambda) p_1 \rho_1^{-1}(\lambda) = \lambda^{q'} p_1 \,, \quad \rho_1(\lambda) p_0 \rho_0^{-1}(\lambda) = \lambda^{q-q'} p_0 \quad \text{for some } q' \,. \tag{3.25}$$

In the same way as quasi-homogeneous superpotentials correspond to conformal field theories in the IR, quasi-homogeneous matrix factorisations correspond to conformal boundary conditions in these CFTs, whereas matrix factorisations which are not quasi-homogeneous undergo an effective RG-flow.[5] We will be mostly interested in quasi-homogeneous matrix factorisations.

[4]Although superpotentials are not renormalised, because of field redefinitions non-quasi-homogeneous superpotentials effectively flow to quasi-homogeneous ones under the RG action.

[5]More details about this can be found in [22].

3.3 B-type defects and matrix factorisations

We will now consider the situation where a Landau-Ginzburg theory with chiral superfields X_i and a superpotential $W_1(X_i)$ is defined on the upper half plane, and a different Landau-Ginzburg theory with superfields Y_i and a superpotential $W_2(Y_i)$ is defined on the lower half plane. The two are separated by a defect on the real line. We would like to describe those defects, which preserve B-type supersymmetry. For this we will indeed follow the same strategy used for the characterisation of B-type boundary conditions in Landau-Ginzburg models reviewed in section 3.2 above.

Again, only B-type supersymmetry preserves the B-type defect line. Exactly as in the boundary case, the B-type supersymmetry variation of the action of the theory on the UHP leads to a boundary term (3.10). The theory on the LHP gives a similar contribution, which however, because of the different relative orientations of the boundary, has opposite sign. Therefore, the total B-type supersymmetry variation of the action of the first Landau-Ginzburg model on the UHP and the second one on the LHP is given by

$$\begin{aligned} \delta_B S &= \delta_B S_D + \delta_B S_F \\ \delta_B S_F &= i \int dx^0 d\theta \big(\bar{\epsilon}(W_1 - W_2) - \epsilon(\bar{W}_1 - \bar{W}_2)\big). \end{aligned} \tag{3.26}$$

Just as in the case of boundaries, $\delta_B S_D$ can be compensated by an appropriate boundary term and $\delta_B S_F$ can be cancelled by introducing additional fermionic degrees of freedom on the defect. The same reasoning as outlined in section 3.2 for the case of boundary conditions leads to the conclusion that B-type defects between the two Landau-Ginzburg models are characterised by matrix factorisations of the difference $W = W_1 - W_2$ of the respective superpotentials. As in the boundary case such a matrix factorisation gives rise to a defect contribution Q_{def} to the BRST-charge, which squares to W. Also the discussion of equivalence and trivial matrix factorisations carries over directly from the discussion of boundary conditions. Moreover, given two matrix factorisations P, P' of W, in the same way as for boundary conditions, the cohomology $\mathcal{H}(P, P')$ of the BRST-operator induced by Q_{def} represents the space of topological defect changing fields (topological closed strings twisted by the two defects). Note however that defects carry more structure than boundary conditions. Unlike boundaries, defects can form junctions where more than two defects meet, and there are fields localised on these junctions (topological closed strings twisted by more than two defects). We will come back to this point later.

Before discussing more of the structure of defects in Landau-Ginzburg models, we would like to remark that the conclusion that B-type defects between two Landau-Ginzburg models are characterised by matrix factorisation of the difference of their superpotentials is indeed consistent with the folding trick. As alluded to in section 2, the folding trick relates defects between two two-dimensional theories $\mathcal{C}_1$ and $\mathcal{C}_2$ to boundary conditions in the product theory $\mathcal{C}_1 \otimes \overline{\mathcal{C}}_2$, where $\overline{\mathcal{C}}_2$ is the theory $\mathcal{C}_2$ with left and right moving sectors interchanged. Folding the Landau-Ginzburg model with superpotential W_2 from the LHP to the UHP maps $x^\pm \mapsto x^\mp$ and likewise $\theta^\pm \mapsto \theta^\mp$. In particular, it maps the D-term of the theory on the LHP to the corresponding D-term on the UHP, while the F-term changes

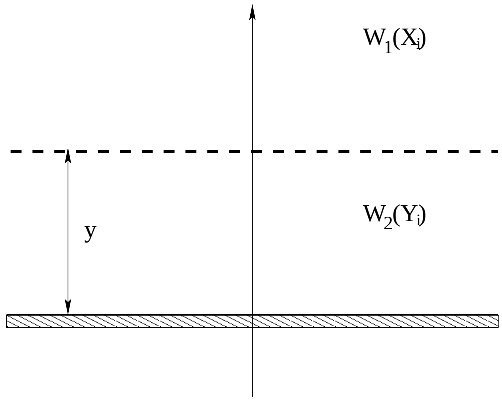

Figure 1: Landau-Ginzburg models with superpotentials W_1 and W_2 on the upper half plane separated by a defect (dashed line). Taking the defect to the boundary ($y \to 0$), one obtains a new boundary condition on the real line.

sign[6]

$$\begin{aligned} S_{\rm LHP} &= \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dt d^4\theta K(Y_i, \bar{Y}_i) + \int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dt \int d\theta^+ d\theta^- W_2(Y_i) \qquad (3.27) \\ &\mapsto \int_{0}^{\infty} dx \int_{-\infty}^{\infty} dt d^4\theta K(Y_i, \bar{Y}_i) - \int_{0}^{\infty} dx \int_{-\infty}^{\infty} dt \int d\theta^+ d\theta^- W_2(Y_i) \ . \end{aligned}$$

The theory on the UHP obtained after folding up the W_2-Landau-Ginzburg model from the LHP is the Landau-Ginzburg model with chiral superfields X_i and Y_i whose Kähler potential is just the sum of the Kähler potentials of the individual models, while its superpotential is the difference $W = W_1 - W_2$ of their superpotentials. As discussed in section 3.2, B-type boundary conditions in this model are indeed characterised by matrix factorisations of W, which according to the folding trick then carries over to B-type defects between Landau-Ginzburg models with superpotentials W_1 and W_2. Thus, the folding trick provides an alternative derivation for the fact that B-type defects between Landau-Ginzburg models can be described by matrix factorisations of the difference of their superpotentials.

Note that if W_1 and W_2 are quasi-homogeneous with respect to some $\mathbb{C}^*$-action, then so is $W_1(X_i) - W_2(Y_i)$. As for boundary conditions, the corresponding quasi-homogeneous matrix factorisations give rise to conformal defects in the IR CFT.

4. Defect operation in Landau-Ginzburg models

Having identified B-type defects between Landau-Ginzburg models as matrix factorisations, one can make use of this rather elegant description to study properties of these defects. For instance, one can investigate situations in which both defects and boundaries, or in which various defects are present.

[6]The measure $d^4\theta$ appearing in the D-term is parity invariant, whereas the measure $d\theta^+ d\theta^-$ in the integral over chiral superspace changes sign.

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As discussed in section 2, upon topological twisting, defects preserving the appropriate supersymmetries become topological. This means that correlation functions in the presence of such defects in the topologically twisted theory do not change when the defects are shifted on the world sheet. In particular, one can bring defects on top of each other or onto world sheet boundaries. Note that for generic (even supersymmetric) defects in the untwisted theory this is not possible. Only purely transmissive defects can be shifted in the untwisted models, and correlation functions diverge when two non-topological defects approach each other, or such a defect approaches a world sheet boundary. These singularities however disappear upon topological twisting. Therefore, in the B-twisted theory one can bring two B-type defects together to obtain another one, and one can bring a B-type defect on top of a boundary satisfying B-type boundary conditions to obtain a new boundary condition. That means, B-type defects can be composed and act on B-type boundary conditions in the twisted models. It is this composition of B-type defects in Landau-Ginzburg models and their action on B-type boundary conditions which we would like to describe in this section.

4.1 Composition of defects and action on boundary conditions

Let us start with the action of defects on boundary conditions. For this consider a theory on the upper half plane consisting of a Landau-Ginzburg model with chiral superfields X_i and superpotential $W_1(X_i)$ defined on the domain $\mathbb{R}+i\mathbb{R}^{>y}$, and a Landau-Ginzburg model with chiral superfields Y_i and superpotential $W_2(Y_i)$ on the domain $\mathbb{R}+iy$ (c.f. figure 1). The two domains are separated by a B-type defect at $\mathbb{R}+iy$ defined by a matrix factorisation of $W(X_i,Y_i) = W_1(X_i) - W_2(Y_i)$, and we impose B-type boundary conditions on $\mathbb{R}$ specified by a matrix factorisation of $W_2(Y_i)$. Let us denote the respective defect and boundary BRST-charges by $Q_{\rm def}$ and $Q_{\rm bd}$. They satisfy $Q_{\rm def}^2 = (W_1 - W_2)$, $Q_{\rm bd}^2 = W_2$. What happens when the defect is moved onto the boundary, i.e. when $y \to 0$ is that in the limit, both defect and boundary fermions $\pi_i^{\rm def}$, $\bar\pi_i^{\rm def}$ and $\pi_i^{\rm bd}$, $\bar\pi_i^{\rm bd}$ together with $Q_{\rm def}$ and $Q_{\rm bd}$ are now defined on the world sheet boundary $\mathbb{R}$. The new boundary condition on $\mathbb{R}$ created by moving the defect on top of the original boundary condition has boundary BRST-charge

$$Q'_{\rm bd} = Q_{\rm def} + Q_{\rm bd}\,. \tag{4.1}$$

Since $Q_{\rm bd}$ and $Q_{\rm def}$ anti-commute,

$$\left(Q'_{\rm bd}\right)^2 = Q_{\rm def}^2 + Q_{\rm bd}^2 = W_1(X_i) - W_2(Y_i) + W_2(Y_i) = W_1(X_i) \tag{4.2}$$

and therefore $Q'_{\rm bd}$ is indeed a BRST-charge of a B-type boundary condition in a Landau-Ginzburg model with superpotential W_1. Note however that $Q'_{\rm bd}$ still involves the chiral superfields Y_i of the Landau-Ginzburg model squeezed in between defect and boundary. In the limit, they are promoted to new boundary degrees of freedom.

In terms of matrix factorisations this can be formulated as follows. Let

$$P: \quad P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0\,, \qquad p_1p_0 = (W_1(X_i) - W_2(Y_i))\mathrm{id}_{P_0}\,, \quad p_0p_1 = (W_1(X_i) - W_2(Y_i))\mathrm{id}_{P_1} \tag{4.3}$$

be the matrix factorisation of $W(X_i, Y_i) = W_1(X_i) - W_2(Y_i)$ representing the defect at $\mathbb{R} + iy$, and let the original boundary condition on $\mathbb{R}$ correspond to the matrix factorisation

$$Q: \quad Q_1 \underset{q_0}{\overset{q_1}{\rightleftarrows}} Q_0\,, \qquad q_1 q_0 = W_2(Y_i)\mathrm{id}_{Q_0}\,, \quad q_0 q_1 = W_2(Y_i)\mathrm{id}_{Q_1} \tag{4.4}$$

of $W_2(Y_i)$. The new boundary condition arising on $\mathbb{R}$ in the limit $y \to 0$ is given by the *tensor product matrix factorisation*

$$\begin{aligned} Q':\ Q_1' = \big(P_1 \otimes_{\mathbb{C}[Y_i]} Q_0\big) \oplus \big(P_0 \otimes_{\mathbb{C}[Y_i]} Q_1\big) \underset{q_0'}{\overset{q_1'}{\rightleftarrows}} Q_0' = \big(P_0 \otimes_{\mathbb{C}[Y_i]} Q_0\big) \oplus \big(P_1 \otimes_{\mathbb{C}[Y_i]} Q_1\big) \\ \text{with} \qquad q_1' = \begin{pmatrix} p_1 & -q_1 \\ q_0 & p_0 \end{pmatrix}, \qquad q_0' = \begin{pmatrix} p_0 & q_1 \\ -q_0 & p_1 \end{pmatrix}. \end{aligned} \tag{4.5}$$

Since Q' represents a B-type boundary condition in the Landau-Ginzburg model with chiral superfields X_i and superpotential $W_1(X_i)$, it has to be regarded as a matrix factorisation over $\mathbb{C}[X_i]$. However, by construction, the Q_i' are really free $\mathbb{C}[X_i, Y_i]$-modules, therefore in particular free $\mathbb{C}[X_i]$-modules of infinite rank, which means that the matrix factorisation Q' defined by (4.5) is a matrix factorisation of infinite rank over $\mathbb{C}[X_i]$.

Thus, moving a B-type defect on top of a B-type boundary, one obtains a boundary condition defined by a matrix factorisation of infinite rank. This is due to the new boundary degrees of freedom arising from the bulk fields Y_i of the Landau-Ginzburg model squeezed in between boundary and defect.

As it turns out, this is only an artifact of the construction. The matrix factorisations (4.5) obtained from finite rank matrix factorisations P and Q are always equivalent up to trivial matrix factorisations to finite rank matrix factorisations of $W_1(X_i)$. That $Q' = P \otimes Q$ is of infinite rank is entirely due to the appearance of spurious trivial matrix factorisations (brane-anti-brane pairs) which are physically irrelevant. Extracting the reduced finite rank matrix factorisation from it is the non-trivial part of the analysis of the action of B-type defects on B-type boundary conditions in topological Landau-Ginzburg models. In section 4.2 below we will present an argument why the matrix factorisations Q' can always be reduced to finite rank, and we will discuss a method to extract the reduced matrix factorisations. Explicit examples will be analysed in section 6.

Before turning to a discussion of the composition of B-type defects, let us remark that the representation of the action of B-type defects on B-type boundary conditions in terms of the tensor product (4.5) is also very natural from the point of view of the topological spectra. As explained in section 3.3 the topological defect changing spectra between two B-type defects represented by matrix factorisations P' and P of $W = W_1 - W_2$ is given by the BRST-cohomology $\mathcal{H}^*(P', P)$.

Now, for P' one can in particular choose a tensor product $P' = R \otimes \bar{Q}$ of matrix factorisations R of W_1 and $\bar{Q}$ of $-W_2$. (Given a matrix factorisation Q of W_2, we denote by $\bar{Q}$ the matrix factorisation of $-W_2$ obtained by $q_1 \mapsto -q_1$.) Such a tensor product matrix factorisation in fact represents a purely reflexive defect, i.e. a tensor product of boundary conditions in the two Landau-Ginzburg models. Therefore, the topological cylinder amplitude with defects corresponding to P' and P inserted along the cylinder, is in fact nothing

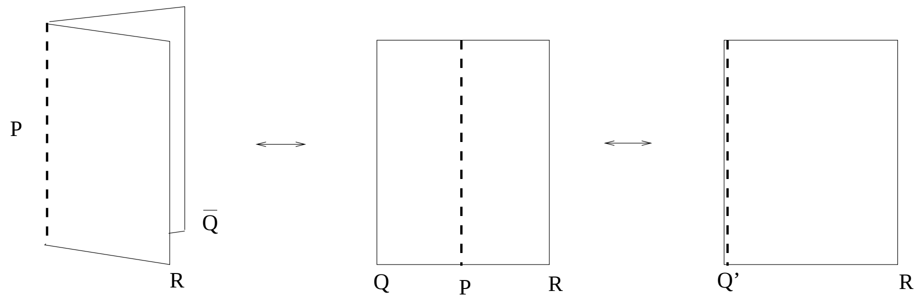

Figure 2: From left to right: 1) The configuration with defects P and $P' = \bar{Q} \otimes R$, 2) open strings between boundary conditions R and Q, twisted by the defect P, 3) As a limit of 2) one obtains open strings between a new boundary condition $Q' = Q \otimes P$ and R.

but the topological amplitude on a strip with boundary conditions corresponding to Q and R along the boundaries, and a defect corresponding to P inserted between them. Hence, for $P' = R \otimes \bar{Q}$ the spectrum $\mathcal{H}^*(P', P)$ in fact also represents the spectrum of topological open strings between the D-branes corresponding to R and Q twisted by the defect corresponding to P. Moreover, since in the topologically twisted theory B-type defects are topological, the spectrum should not change when moving the defect. In particular, it should not change, when bringing the defect on top of one of the boundaries, the one corresponding to Q say. From this it follows that the spectrum $\mathcal{H}^*(P', P)$ should indeed also describe the spectrum of topological open strings between the D-brane corresponding to R on one side and the D-brane described by Q' arising from bringing the defect P onto the boundary condition Q on the other:

$$\mathcal{H}^*(P' = R \otimes \bar{Q}, P) \cong \mathcal{H}^*(R, Q') \,. \tag{4.6}$$

But from the construction of the BRST-operator, it is easy to see that indeed

$$\mathcal{H}^*(R \otimes \bar{Q}, P) \cong \mathcal{H}^*(R, P \otimes Q) \,, \tag{4.7}$$

which shows that the tensor product factorisation $Q' = P \otimes Q$ has the spectrum expected from the matrix factorisation representing the boundary condition obtained by moving the defect described by P onto the boundary with boundary condition Q.

Completely analogously to the action of B-type defects on B-type boundary conditions one can describe the action of them on other B-type defects, i.e. their composition. On the level of matrix factorisation, the latter is also represented by taking the tensor product of the matrix factorisations representing the defects which are being composed. For this replace in the discussion above the matrix factorisation (4.4) by a matrix factorisation of $W_2(Y_i) - W_3(Z_i)$ over $\mathbb{C}[Y_i, Z_i]$ representing a defect between the Landau-Ginzburg models with superpotentials $W_2(Y_i)$ and $W_3(Z_i)$ respectively. The tensor product (4.5) then gives rise to an infinite rank matrix factorisation of $W_1(X_i) - W_3(Z_i)$ representing the defect emerging as the composition of the two defects. As in the case of the action on boundary conditions it is in fact equivalent modulo trivial matrix factorisations to a finite rank one, and also for the analysis of the composition of defects the challenge lies in the reduction of the infinite rank tensor product matrix factorisation to finite rank.

We would like to close the general discussion of composition of B-type defects in Landau-Ginzburg models and their action on B-type boundary conditions with the following remark. As pointed out in section 3.3, one fundamental difference between boundary conditions and defects is the possibility of the latter to form junctions, which also carry fields. The discussion above in fact suggests a simple method to calculate the topological spectra of fields localised on the junctions formed by n B-type defects in Landau-Ginzburg models. Namely, let $W_1, \ldots, W_n$, $W_{n+1} = W_1$ be superpotentials, and for $1 \leq i \leq n$ let P^i be matrix factorisations of $W_i - W_{i+1}$ representing B-type defects between the respective Landau-Ginzburg models. To calculate the topological spectrum $\mathcal{H}^*(P^1, \ldots, P^n)$ of fields on the junction formed by these defects (topological closed strings twisted by all of them) we note that as above, the topological spectra should not change when shifting the defects. So in particular, we can bring the last $n-1$ of them on top of each other, and the spectrum of fields on the junction is identical to the spectrum of defect changing fields between the defect represented by P^1 and the one obtained by composing the defects associated to P^i with $i > 1$. Since the latter is represented by the matrix factorisation $P^2 \otimes \ldots \otimes P^n$ one obtains

$$\mathcal{H}^*(P^1, \ldots, P^n) \cong \mathcal{H}^*(\bar{P}^1, P^2 \otimes \ldots \otimes P^n). \tag{4.8}$$

4.2 Defect operation and matrix factorisations

In the section 3.3 we have argued that similarly to B-type boundary conditions, also B-type defects in Landau-Ginzburg models can be described by means of matrix factorisations. We have explained that in this formulation, the action of these defects on B-type boundary conditions and defects has a simple realisation in terms of the tensor product (4.5) of the respective matrix factorisations. As was pointed out in the previous section, the tensor product matrix factorisations obtained in this way are of infinite rank however. Here we will argue that they indeed are always equivalent to matrix factorisations of finite rank. That means, it is always possible to reduce them to matrix factorisations of finite rank by splitting off infinitely many trivial matrix factorisations. It is this reduction which is the non-trivial part in the analysis of the action of B-type defects, and we will discuss a method to deal with it below. We will focus on the action of B-type defects on B-type boundary conditions, but the discussion of the composition of defects works exactly analogously.

The basic idea we employ to show that the tensor product matrix factorisations obtained are equivalent to finite rank factorisations is to identify the reduced rank as the dimension of a certain BRST-cohomology group, which can be calculated directly from the infinite rank representative. (In a geometric context, one would want to count the bosonic open strings between the D-brane under consideration and the basic D-brane with Neumann boundary conditions in all directions, carrying only one type of charge.) This argument only works in the case that W and the matrix factorisations under consideration are quasi-homogeneous.[7] Since we are mostly interested in quasi-homogeneous matrix fac-

[7] Indeed it also works for non-homogeneous matrix factorisations defined over Laurent rings instead of polynomial rings.

torisations, we will restrict the discussion to this case, but we believe that the statement also holds in the general situation.

Let us start the discussion by the following remark. Consider a matrix factorisation

$$Q: \quad Q_1 \underset{q_0}{\overset{q_1}{\rightleftarrows}} Q_0\,, \qquad q_1 q_0 = W(X_i)\mathrm{id}_{Q_0}\,, \quad q_0 q_1 = W(X_i)\mathrm{id}_{Q_1} \tag{4.9}$$

of $W(X_i)$ over the ring $\mathbb{C}[X_i]$. Now suppose that q_1 or q_0 have an entry which is a unit (i.e. an invertible element) in $\mathbb{C}[X_i]$. It is easy to see that in this case there is an equivalence $(u_i, v_i = u_i^{-1})$ as in (3.20) which brings Q into the form $Q \cong Q' \oplus T$, where T is the trivial matrix factorisation (3.22). In particular, Q can be reduced to $Q \cong Q'$. This can be done until there are no more unit entries in the matrix factorisation, in which case no trivial matrix factorisation can be split off anymore in this way. Let us assume this to be true for the matrix factorisation Q and let N be its rank. Under these circumstances, the rank of Q can be calculated as the dimension of the BRST-cohomology $\mathcal{H}^0(Q,S)$

$$\mathrm{rank}(Q) = \dim \mathcal{H}^0(Q,S)\,, \tag{4.10}$$

where S is the tensor product[8] of the rank-one factorisations P^i defined by $p_1^i = X_i$, $p_0^i = B_i(X_i)$ with $W = \sum_i X_i B_i$. This can be seen as follows. The Koszul resolution of the module $M = \mathbb{C}[X_i]/(X_i)$ regarded as a $\mathbb{C}[X_i]$-module can be used to construct an $R := \mathbb{C}[X_i]/(W)$-free resolution of M, which after l steps turns into the two-periodic resolution of $\mathrm{coker}(p_1)$ defined by p_1 and p_0. Here, $l = n$ is the number of variables X_i if n is even, and $l = n-1$ if n is odd. Thus,

$$\mathrm{Ext}^i_R(\cdot, \mathrm{coker}(p_1)) \cong \mathrm{Ext}^{i+l}_R(\cdot, M)\,. \tag{4.11}$$

This has been discussed in detail in section 4.3 of [23]. Using the general fact [23] that for all $i > 0$

$$\mathcal{H}^0(P,Q) \cong \mathrm{Ext}^{2i}_R(\mathrm{coker}(p_1), \mathrm{coker}(q_1))\,,\ \mathcal{H}^1(P,Q) \cong \mathrm{Ext}^{2i-1}_R(\mathrm{coker}(p_1), \mathrm{coker}(q_1))\,,$$

one obtains

$$\mathcal{H}^0(Q,S) \cong \mathrm{Ext}^{2+l}_R(\mathrm{coker}(q_1), M)\,. \tag{4.12}$$

This Ext-group can be calculated by means of the two-periodic resolution

$$\ldots \xrightarrow{q_1} R^N \xrightarrow{q_0} R^N \xrightarrow{q_1} R^N \longrightarrow \mathrm{coker}(q_1) \longrightarrow 0\,. \tag{4.13}$$

Namely, it is given by the cohomology of the complex obtained by applying the functor $\mathrm{Hom}_R(\cdot, M)$ to the resolution (4.13). Since the q_i (in a certain basis) only have non-unit homogeneous entries, the differentials of this complex all vanish, and therefore the cohomology in every degree is given by M^N. In particular, the dimension of the cohomology groups is N, the rank of Q.

[8] As in the definition of D_g in (5.9) below, the rank-one factors are not necessarily unique, but the resulting tensor product matrix factorisation is up to equivalence.

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But now, $\dim \mathcal{H}^0(Q,S)$ does not change when one adds trivial matrix factorisations to Q. This implies that $\dim \mathcal{H}^0(Q,S)$ indeed calculates the reduced rank of the matrix factorisation Q, i.e. the rank of the matrix factorisation obtained from Q by splitting off all trivial matrix factorisations in the way described above.[9]

We will use this to show that the reduced rank of tensor product matrix factorisations representing the boundary conditions obtained by applying a B-type defect to a B-type boundary condition is always finite (assuming that the factor matrix factorisations are of finite rank). So let P as in (4.3) represent a B-type defect between two Landau-Ginzburg models, Q as in (4.4) a B-type boundary condition in one of them, and Q' defined in (4.5) their tensor product. To show that the reduced rank of Q' is finite we again make use of (4.12). As above we use the resolution (4.13) for Q' to compute the Ext-groups, which are then given by the cohomology of the sequence

$$\ldots \xrightarrow{\widetilde{q}_0'} (M')^{N'} \xrightarrow{\widetilde{q}_1'} (M')^{N'} \xrightarrow{\widetilde{q}_0'} (M')^{N'} \xrightarrow{\widetilde{q}_1'} \ldots \tag{4.14}$$

Here N' denotes the rank of Q', $M' = \mathbb{C}[X_i, Y_i]/(X_i)$ and $\widetilde{q}_i'$ are obtained from the q_i' by setting $X_i = 0$. But similarly as in the discussion of (4.7), one recognises this complex as the one computing $\mathcal{H}(Q, \overline{\widetilde{P}})$, where $\widetilde{P}$ is the matrix factorisation over $\mathbb{C}[Y_i]$ obtained from P by setting $X_i = 0$. It is in particular a finite rank matrix factorisation of $W_2(Y_i)$. We therefore obtain

$$\mathcal{H}^0(Q' = P \otimes Q, S) \cong \mathcal{H}^i(Q, \overline{\widetilde{P}})\,, \tag{4.15}$$

for some i, where the latter is the BRST-cohomology between two finite rank matrix factorisations of $W_2(Y_i)$, which in particular is finite dimensional. Hence the reduced rank of Q' is finite.

Having established that Q' can be reduced to a matrix factorisation of finite rank, we would now like to comment on how to obtain a reduced form. To find the explicit equivalence on the level of matrix factorisations is difficult in general. (A very simple example is discussed in appendix B.) On the level of modules, what one has to do is to regard $\mathrm{coker}(q_1')$ as an R-module and split off all free summands. A trick, which will prove useful in the examples presented in section 6 is the following. Instead of analysing $\mathrm{coker}(q_1')$ one can consider the module $V = \mathrm{coker}(p_1 \otimes \mathrm{id}_{Q_0}, -\mathrm{id}_{P_0} \otimes q_1)$. This module has the R-free resolution

$$\ldots \xrightarrow{q_1'} Q_0' \xrightarrow{q_0'} Q_1' \xrightarrow{q_1'} Q_0' \xrightarrow{q_0'} Q_1' \xrightarrow{(p_1 \otimes \mathrm{id}_{Q_0}, -\mathrm{id}_{P_0} \otimes q_1)} P_0 \otimes Q_0 \longrightarrow V \longrightarrow 0, \tag{4.16}$$

which after two steps turns into the R-free resolution of $\mathrm{coker}(q_1')$ obtained from the matrix factorisation Q'. Therefore, instead of reducing $\mathrm{coker}(q_1')$, we can just as well reduce V and take the matrix factorisation which can be obtained from a resolution of the reduced module by chopping off the first two terms. Indeed, in the examples presented in section 6 this trick will prove to be very useful, because V itself will already be of finite rank.

Completely analogously to the action of B-type defects on B-type boundary conditions, the composition of B-type defects can be described. For this replace the matrix

[9] A priori it might be possible that Q is equivalent to a matrix factorisation with even smaller rank.

factorisation (4.4) with a matrix factorisation of $W_2(Y_i) - W_3(Z_i)$ over $\mathbb{C}[Y_i, Z_i]$ representing a defect between the Landau-Ginzburg model with superpotential $W_2(Y_i)$ and the one with superpotential $W_3(Z_i)$. The tensor product (4.5) then gives rise to an infinite rank matrix factorisation of $W_1(X_i) - W_3(Z_i)$, which as in the case of the action on boundary conditions is in fact equivalent to a finite dimensional one. Thus, from B-type defects between Landau-Ginzburg theories with superpotentials $W_1(X_i)$ and $W_2(Y_i)$, and $W_2(Y_i)$ and $W_3(Z_i)$ respectively, one obtains one between the Landau-Ginzburg theories with superpotentials $W_1(X_i)$ and $W_3(Z_i)$.

5. Symmetry defects

If a two-dimensional field theory exhibits symmetries, i.e. automorphisms of its Hilbert space which commute with energy and momentum operators, then these give rise to topological defects. The corresponding defect operators are simply given by the automorphisms themselves, and the closed string sectors twisted by such defects are the ordinary twisted sectors known from orbifold constructions.[10] Obviously, these defects compose according to the symmetry group of the theory, and in particular every such defect has an inverse. They are group-like defects as discussed in [6].

For a Landau-Ginzburg model with chiral superfields X_i and superpotential W there is a simple class of symmetries, whose action is defined by linear and unitary[11] action on the superfields X_i:

$$X_i \mapsto g(X_i)\,, \quad \text{such that } W(g(X_i)) = W(X_i)\,. \tag{5.1}$$

We will denote the group of these transformations by Γ. The corresponding defects can easily be described by means of gluing conditions of the chiral superfields along the defect. Let us consider a Landau-Ginzburg model with superpotential W on the full plane with such a defect along the real line. Denote by X_i and Y_i the chiral superfields on the UHP and the LHP respectively. Then for every $g \in \Gamma$ as above, one can define a defect D_g by imposing the gluing conditions

$$(X_i(x+iy) - g(Y_i)(x-iy)) \to 0 \quad \text{for} \quad y \to 0 \tag{5.2}$$

on the chiral superfields on the UHP and LHP along the real line. Obviously, these gluing conditions cancel the supersymmetry variation (3.26) of the bulk F-term in the presence of the defect without the introduction of any additional degrees of freedom. Moreover, these defects are also compatible with A-type supersymmetry. To see this, consider the full supersymmetry variation (3.5) of the F-term of the theory on the upper half plane. The result is

$$\delta S = i \int_{\mathbb{R}} dx^0 \big(\bar{\epsilon}_+ \omega_+^{(1)} - \bar{\epsilon}_- \omega_-^{(1)} + \epsilon_- \bar{\omega}_-^{(1)} - \epsilon_+ \bar{\omega_+}^{(1)}\big). \tag{5.3}$$

where we have expanded the chiral superfield W_1 as

$$W_1 = w^{(1)}(y^\pm) + \theta^\alpha \omega_\alpha^{(1)}(y^\pm) + \theta^+\theta^- F^{(1)}(y^\pm), \tag{5.4}$$

[10] In fact, also dualities between different theories can give rise to such defects.

[11] The standard Kähler potential $K = \sum_i \bar{X}_i X_i$ has to be invariant.

This can be compensated by the variation of a theory defined on the lower half plane if

$$
\begin{aligned}
(\omega_\pm^{(1)}(x+iy) \,-\, \omega_\pm^{(2)}(x-iy)) &\rightarrow 0\,, \\
(\bar\omega_\pm^{(1)}(x+iy) \,-\, \bar\omega_\pm^{(2)}(x-iy)) &\rightarrow 0
\end{aligned}
\tag{5.5}
$$

modulo total derivatives in the limit $y \rightarrow 0$. As one easily checks, these conditions are satisfied in the case that the chiral superfields obey the gluing relations (5.2). The latter furthermore imply gluing conditions

$$
Q_\pm^{(1)} = Q_\pm^{(2)}\,, \quad \bar{Q}_\pm^{(1)} = \bar{Q}_\pm^{(2)} \tag{5.6}
$$

for the Landau-Ginzburg supercharges (3.6) along the defect line, which ensures that indeed the full $N=(2,2)$ supersymmetry is preserved.

Since the defects are topological, one can compose them with the obvious result

$$
D_g D_{g'} = D_{gg'}\,. \tag{5.7}
$$

As mentioned above the defect spectra obtained from defects defined by group actions are nothing but the twisted spectra usually discussed in the context of the corresponding orbifold models. We refer to [24, 25], for a discussion of the twisted sectors in Landau-Ginzburg orbifolds.

Even though these defects have a very nice and simple description not involving new degrees of freedom on the defect, we would like to make contact with the discussion of the previous sections and show how to formulate them in terms of matrix factorisations. Here, we can take inspiration from a similar discussion in the context of boundary conditions. B-type boundary conditions for Landau-Ginzburg models had first been introduced in [26, 18] without the introduction of additional boundary degrees of freedom. After the discovery that matrix factorisations provide more general boundary conditions, it was proposed in [27, 28] that the original boundary conditions of [26, 18] can indeed be realised as matrix factorisations, having one linear factor representing the gluing conditions of the chiral fields along the boundary. This suggests that the group like defects discussed above should be realised as linear matrix factorisations as well. Indeed, for every g as above $W(X_i)-W(Y_i)$ can be factorised as[12]

$$
W(X_i) - W(Y_i) = \sum_j (X_j - g(Y_j)) A_j(X_i, Y_i)\,, \tag{5.8}
$$

for some polynomials $A_j(X_i, Y_i)$, generalising the prescription for $g=1$ in [29]. We propose that the defects D_g can then be represented by the tensor product matrix factorisations

$$
D_g = \bigotimes_i P^i \tag{5.9}
$$

of the rank-one factorisations defined by

$$
p_1^i = (X_i - g(Y_i))\,, \quad p_0^i = A_i(X_i, Y_i)\,. \tag{5.10}
$$

[12]This factorisation may not be unique, but the matrix factorisations (5.9) resulting from different choices of A_j in (5.8) are equivalent.

Let us gather some evidence for this proposal. It is indeed very easy to verify that the matrix factorisations (5.9) lead to the desired action on B-type defects and boundary conditions. To see this, consider a matrix factorisation D_g as defined above and a matrix factorisation Q which is either a matrix factorisation of $W(Y_i)$ corresponding to a B-type boundary condition, or a matrix factorisation of $W(Y_i) - W(Z_i)$ representing another defect. We set $R := \mathbb{C}[X_i]/(W)$ in the first, $R := \mathbb{C}[X_i, Z_i]/(W(X_i) - W(Z_i))$ in the second case.

The result of the action of D_g on Q is given by the matrix factorisation $D_g \otimes Q$. To reduce this, we employ the same trick used to get (4.12). Namely, the module

$$M := \mathrm{coker}((X_1 - g(Y_1))\mathrm{id}_{Q_0}, \ldots, (X_n - g(Y_n))\mathrm{id}_{Q_0}, q_1) \tag{5.11}$$

has an R-free resolution which after $l = n+1$ for n odd, and $l = n$ for n even steps turns into the matrix factorisation $D_g \otimes Q$. (This resolution is related to the Koszul complex, and is discussed in a similar context in section 4.3 of [23].) Therefore the matrix factorisation $D_g \otimes Q$ is equivalent to the matrix factorisation into which the R-free resolution of M turns after l steps. But M is nothing else than

$$M \cong \mathrm{coker}(q_1(Y_i = g^{-1}(X_i))) \tag{5.12}$$

which obviously has a completely two-periodic resolution, namely the matrix factorisation $Q(Y_i = g^{-1}(X_i))$ over R. Thus, D_g acts on matrix factorisations by setting $Y_i = g^{-1}(X_i)$. In particular one obtains the desired composition of the symmetry defects D_g, because $D_g \otimes D_{g'}$ is equivalent to $D_{gg'}$.

Also the analysis of defect spectra supports the identification of the matrix factorisations (5.9) with group like defects. The spectra associated to the symmetry defects do indeed agree with the spectra of bulk fields twisted by the respective group elements as calculated in [25, 24]. More precisely, one can show that the defect spectra $\mathcal{H}^*(D_g, D_1)$ are isomorphic to the g-twisted bulk Hilbert spaces.[13] For instance, in the case of a Landau-Ginzburg model with a single chiral superfield X it is indeed very easy to see by direct calculation that $\mathcal{H}^*(D_1, D_1)$ is purely bosonic and isomorphic to the bulk chiral ring $\mathbb{C}[X]/(\partial W)$, i.e. the untwisted bulk Hilbert space. For $g \neq 1$, on the other hand there are no bosons in the defect spectra $\mathcal{H}^*(D_g, D_1)$, and only a single fermion ω, corresponding to the unique ground state in the g-twisted sector of the orbifold.

This easily generalises to tensor products of this situation (in particular the g act diagonally on the X_i), in which case each tensor factor contributes to $\mathcal{H}^*(D_g, D_1)$ either polynomials in $\mathbb{C}[X_i]/(\partial_i W)$ in case X_i is g-invariant, or a fermion ω_i, if it is not. Hence, $\mathcal{H}^*(D_g, D_1)$ is spanned by polynomials in g-invariant variables multiplied by one fermion for each variable which is not g-invariant. This can be written as

$$\mathcal{H}^*(D_g, D_1) \cong \mathbb{C}[X_i^{g-\mathrm{inv}}]/(\partial_i(W_{g-\mathrm{inv}})) \prod_{g(X_j) \neq X_j} \omega_j \,, \tag{5.13}$$

with $W_{g-\mathrm{inv}}$ the polynomial obtained from W by setting all non-g-invariant variables to zero. It is easily recognised as the g-twisted orbifold sector obtained in [24, 25].

[13] By means of the composition of the D_g discussed above $\mathcal{H}^*(D_{g'}, D_g) \cong \mathcal{H}^*(D_{g'g^{-1}}, D_1)$.

Indeed, the statement that the defect spectra $\mathcal{H}^*(D_g, D_1)$ are isomorphic to the g-twisted sectors is true in the general situation. The proof for the general case is presented in appendix A.[14]

6. Defects in minimal models

Up to now we have discussed symmetry defects in arbitrary Landau-Ginzburg models. The matrix factorisations describing these defects, their action on B-type boundary conditions and their composition properties have been discussed in section 5. Here, we would like to discuss more general defects in Landau-Ginzburg models with one chiral superfield and superpotential $W(X) = X^d$, and in their closely related cousins, theories with one additional superfield and superpotential $W(X) = X^d + Z^2$. The bulk chiral rings of these two theories are equivalent, but there are differences in the D-brane spectra, as discussed in [15]. Indeed the two theories can be regarded as $\mathbb{Z}_2$ orbifolds of each other, and therefore, adding a further square leads again to the initial theory. On the level of matrix factorisation this property is known as Knörrer periodicity.

In the IR these models become respectively $N = 2$ superconformal minimal models and $\mathbb{Z}_2$-orbifolds thereof. Both these models share the same Hilbert space, but differ in the action of $(-1)^F$. They are well understood conformal field theories in which conformal defects can be explicitly studied.

In 6.1, we will construct and analyse defects within the Landau-Ginzburg framework presented in the previous sections. In 6.2 we will make contact with the CFT-analysis. We will restrict our attention to defects between one and the same Landau-Ginzburg model, in which case constructions for the corresponding conformal defects are known. The case of defects between Landau-Ginzburg models with different superpotentials will be investigated in [30].

6.1 Landau-Ginzburg approach

Let us start with the case $W(X) = X^d$. The model with superpotential $W(X,Z) = X^d+Z^2$ will be discussed later in Subsection 6.1.4. There are certain obvious candidates for defect matrix factorisations of $W(X) - W(Y) = X^d - Y^d$. On the one hand, these are the tensor product matrix factorisations

$$T_{i,j}: \quad t_1^{i,j} = \begin{pmatrix} X^i & Y^j \\ Y^{d-j} & X^{d-i} \end{pmatrix}, \quad t_0^{i,j} = \begin{pmatrix} X^{d-i} & -Y^j \\ -Y^{d-j} & X^i \end{pmatrix}. \tag{6.1}$$

On the other hand there are "permutation type" matrix factorisations of the form

$$P_I^d: \quad p_1^I = \prod_{a\in I}(X - \eta^a Y), \quad p_0^I = \prod_{a\in\{0,\dots,d-1\}-I}(X - \eta^a Y), \tag{6.2}$$

[14]That the spaces $\mathcal{H}^*(D_1, D_1)$ for general W coincide with the bulk-chiral rings has also been observed in [29]

where η is an elementary dth root of unity and I is a strict subset of $\{0, \ldots, d-1\}$. These defects generalise the symmetry defects discussed in section 5. Namely, the Landau-Ginzburg model with superpotential $W = X^d$ allows for the operation of a symmetry group $\mathbb{Z}_d$ on the superfield

$$i \in \mathbb{Z}_d : \quad X \mapsto \eta^i X\,, \tag{6.3}$$

and the corresponding defect matrix factorisations D_i agree with the matrix factorisations $P_{\{i\}}$ of (6.2).

6.1.1 Composition of permutation type matrix defects

The action of the permutation type matrix factorisations $P_{\{i\}}$ has already been discussed in section 5. Here we would like to analyse the composition of defects represented by P_I for arbitrary I. As it will turn out, we will only have to analyse the action of P_I with $|I| = 2$, because successively composing such P_I one can generate all other P_I as well. Considerations will be restricted to the case where I is a set of successive integers modulo d, because these defects have a simple representation in the respective conformal field theories.

As a warm up, let us consider the composition of two defects corresponding to matrix factorisations P_I with $|I| = 2$, $P_{\{m,m+1\}}$ and $P_{\{m',m'+1\}}$. Using the trick described in section 4.2, the result of this composition is the B-type defect represented by the matrix factorisation associated to the $R = \mathbb{C}[X,Z]/(X^d - Z^d)$-module

$$M := \mathbb{C}[X,Y,Z]/((X - \eta^m Y)(X - \eta^{m+1} Y),\ (Y - \eta^{m'} Z)(Y - \eta^{m'+1} Z))\,. \tag{6.4}$$

In M we have the following relations:

$$\begin{aligned} Y^{2+i} - \alpha X Y^{1+i} - \beta X^2 Y^i &= 0 \\ Y^{2+i} - \alpha' Z Y^{1+i} - \beta' Z^2 Y^i &= 0\,, \end{aligned} \tag{6.5}$$

where we abbreviated

$$\begin{aligned} \alpha &:= \eta^{-m} + \eta^{-m-1}\,, \quad \beta := -\eta^{-2m-1}\,, \\ \alpha' &:= \eta^{m'} + \eta^{m'+1}\,, \quad \beta' := -\eta^{2m'+1}\,. \end{aligned} \tag{6.6}$$

From this it follows in particular that the submodules of M built on Y^i for $i \geq 2$ are in fact submodules of those built on 1 and Y, so the task is to understand the latter, i.e. the relations in them. To start note that from (6.5) it follows that

$$\begin{aligned} 0 &= \left(\alpha X - \alpha' Z\right) Y + \left(\beta X^2 - \beta' Z^2\right) \\ &= \left(\alpha X - \alpha' Z\right) \underbrace{\left(Y + \frac{\beta}{\alpha^2}\left(\alpha X + \alpha' Z\right)\right)}_{=:e_1} = 0\,, \end{aligned} \tag{6.7}$$

where use was made of (6.6). In fact, there are no further relations in the submodule built on e_1, so that the latter is just given by ($\alpha'/\alpha = \eta^{m+m'+1}$)

$$R/(X - \eta^{m+m'+1} Z)\,. \tag{6.8}$$

To determine the remaining part of M, we note that (6.5) also gives rise to (6.7) multiplied by Y. Substituting the first of the equations (6.5) into the latter, one obtains, using in particular (6.7) and (6.6)

$$(X-\eta^{m+m'}Z)(X-\eta^{m+m'+1}Z)(X-\eta^{m+m'+2}Z)=0\,, \tag{6.9}$$

which is the only relation in the submodule built on $e_0 := 1 \in M$. Therefore

$$M \cong R/(X-\eta^{m+m'+1}Z) \oplus R/(X-\eta^{m+m'}Z)(X-\eta^{m+m'+1}Z)(X-\eta^{m+m'+2}Z)\,, \tag{6.10}$$

and the defect obtained by composing the defects corresponding to the matrix factorisations $P_{\{m,m+1\}}$ and $P_{\{m',m'+1\}}$ is represented by the sum

$$P_{\{m,m+1\}} * P_{\{m',m'+1\}} = P_{\{m+m'+1\}} \oplus P_{\{m+m',m+m'+1,m+m'+2\}}\,. \tag{6.11}$$

In case $d=3$ the second summand is trivial, if $d>3$, the composition of the two P_I with $|I|=2$ generates a P_I with $|I|=3$.

Let us now consider the more general case, namely the composition of $P_{\{m,m+1\}}$ and $P_{\{m',\dots,m'+a\}}$. The result of this composition is the matrix factorisation associated to the R-module

$$M = \mathbb{C}[X,Y,Z]/\left((X-\eta^m Y)(X-\eta^{m+1}Y), \prod_{i=0}^{a}(Y-\eta^{m'+i}Z)\right)\,. \tag{6.12}$$

As in the special case discussed above, because of the quadratic relation

$$(X-\eta^m Y)(X-\eta^{m+1}Y) = Y^2 - (\eta^{-m}+\eta^{-m-1})XY + \eta^{-2m-1}X^2 = 0\,, \tag{6.13}$$

we only have to consider the submodules built on 1 and Y. To obtain the relations in them, by means of (6.13) we eliminate all Y^i with $i>1$ from

$$F(Y,Z) = \prod_{i=0}^{a}(Y-\eta^{m'+i}Z) = 0 \tag{6.14}$$

to obtain a relation of the form

$$YP(X,Z) + Q(X,Z) = 0\,. \tag{6.15}$$

Multiplying it by Y and again using (6.13) gives rise to another relation

$$\begin{aligned} 0 &= Y^2P(X,Z) + YQ(X,Z) \\ &= Y\left((\eta^{-m}+\eta^{-m-1})XP(X,Z) + Q(X,Z)\right) - \eta^{-2m-1}X^2P(X,Z)\,. \end{aligned} \tag{6.16}$$

Again multiplying by Y one obtains a linear combination of (6.15) and (6.16), thus these two are the only relations on the submodule built on 1 and Y.

Now, by construction

$$F(Y, Z=\eta^{-m'-m-i}X) \sim \prod_{j=-a}^{0}(X-\eta^{m+i-j}Y) \tag{6.17}$$

contains (6.13) as a factor iff $1 \le i \le a$, from which it follows that $P(X,Z)$ and $Q(X,Z)$ have roots $(X - \eta^{m'+m+i}Z)$ for $1 \le i \le a$. Since P and Q have degree a and $a+1$ respectively, it follows that

$$P(X,Z) \sim \prod_{i=1}^{a}(X - \eta^{m'+m+i}Z)\,, \quad \text{and} \quad Q(X,Z) = P(X,Z)q(X,Z)\,, \tag{6.18}$$

where q is a polynomial of degree 1. Therefore relation (6.15) can be written as

$$P(X,Z)(Y + q(X,Z)) = 0\,, \tag{6.19}$$

and using this, relation (6.16) becomes

$$P(X,Z)\underbrace{\left(-\eta^{-2m-1}X^2 - (\eta^{-m} + \eta^{-m-1})Xq(X,Z) - q^2(X,Z)\right)}_{=:S(X,Z)} = 0\,. \tag{6.20}$$

It remains to determine the quadratic polynomial $S(X,Z)$. For this we note that the polynomials $F(Y, Z = \eta^{-m'-m}X)$ and $F(Y, Z = \eta^{-m'-m-a-1}X)$ contain factors $(X - \eta^m Y)$ and $(X - \eta^{m+1}Y)$ respectively. In particular

$$\begin{aligned} 0 &= F(Y, Z = \eta^{-m'-m}X)(X - \eta^{m+1}Y) \\ 0 &= F(Y, Z = \eta^{-m'-m-a-1}X)(X - \eta^{m}Y)\,. \end{aligned} \tag{6.21}$$

Making once again use of the fact that $F = YP + Q$ and the quadratic relation (6.13), one obtains the equations

$$0 = \eta^{-m}XP(X, Z = \eta^{-m'-m}X) + Q(X, Z = \eta^{-m'-m}X) \tag{6.22}$$

$$0 = \eta^{-m-1}XP(X, Z = \eta^{-m'-m-a-1}X) + Q(X, Z = \eta^{-m'-m-a-1}X)\,, \tag{6.23}$$

which can be used to determine the linear polynomial $q(X,Z) = \mu X + \nu Z$. Namely

$$\mu = -\eta^{-m}\frac{1-\eta^a}{1-\eta^{a+1}}\,, \quad \nu = -\eta^{m'+a}\frac{1-\eta}{1-\eta^{a+1}}\,. \tag{6.24}$$

Substituting q in the equation for S, one obtains

$$S(X,Z) \sim (X - \eta^{m+m'}Z)(X - \eta^{m+m'+a+1}Z)\,, \tag{6.25}$$

and hence

$$M \cong \mathbb{C}[X,Z]/\left(\prod_{i=1}^{a}(X - \eta^{m+m'+i})\right) \oplus \mathbb{C}[X,Z]/\left(\prod_{i=0}^{a+1}(X - \eta^{m+m'+i})\right)\,. \tag{6.26}$$

Therefore,

$$P_{\{m,m+1\}} * P_{\{m',\ldots,m'+a\}} = P_{\{m+m'+1,\ldots,m+m'+a\}} \oplus P_{\{m+m',\ldots,m+m'+a+1\}}\,. \tag{6.27}$$

If $d = a-1$ the second summand is trivial, otherwise the action of $P_{\{m,m+1\}}$ on a P_I with $|I| = r$ generates a P_I with $|I| = r+1$, and we see that by composing P_I with $|I| = 2$, we can indeed generate all P_I. Therefore, by means of associativity, (6.27) indeed determines the composition of arbitrary permutation like defects. Using the fusion rules $\mathcal{N}$ of $\widehat{\mathfrak{su}}(2)_{d-2}$ one obtains:

$$P_{\{m_1,\ldots,m_1+l_1\}} * P_{\{m_2,\ldots,m_2+l_2\}} = \bigoplus_{l}\mathcal{N}_{l_1 l_2}^{l}P_{\{\frac{1}{2}(l_1+l_2-l)+m_1+m_2,\ldots,\frac{1}{2}(l_1+l_2+l)+m_1+m_2\}}\,. \tag{6.28}$$

6.1.2 Action of permutation type defects on boundary conditions

Using the results from the previous subsection, also the action of permutation type defects on boundary conditions is completely determined by the action of the defects corresponding to matrix factorisations $P_{\{m,m+1\}}$. So let us investigate the action of such defects on boundary condition represented by matrix factorisations

$$T_a: \quad \mathbb{C}[Y] \underset{Y^{d-a}}{\overset{Y^a}{\rightleftarrows}} \mathbb{C}[Y]\,. \tag{6.29}$$

The resulting boundary condition is given by the matrix factorisation defined by the $R = \mathbb{C}[X]/(W(X))$-module

$$M := \mathbb{C}[X,Y]/((X-\eta^m Y)(X-\eta^{m+1}Y), Y^a)\,. \tag{6.30}$$

The relations on this module are

$$Y^{2+i} - \underbrace{(\eta^{-m}+\eta^{-m-1})}_{=:\alpha} XY^{1+i} + \underbrace{\eta^{-2m-1}}_{=:-\beta} X^2 Y^i = 0\,, \quad Y^a = 0\,, \tag{6.31}$$

and in particular the submodules built on Y^i for $i > 1$ are submodules of the ones built on 1 and Y. Therefore we only have to determine the relations on these two submodules. From (6.31) we obtain

$$\begin{aligned} X^2 Y^{a-1} &= 0 \\ X\left(Y^{a-1} + \frac{\beta}{\alpha} XY^{a-2}\right) &= 0\,, \end{aligned} \tag{6.32}$$

and inductively:

$$\begin{aligned} X^{i+2} Y^{a-i-1} &= 0 \\ X^i \left(Y^{a-i} - \frac{\sum_{j=0}^{i-1}\eta^{j-m}}{\sum_{j=0}^{i}\eta^j} XY^{a-i-1}\right) &= 0\,. \end{aligned} \tag{6.33}$$

In particular:

$$\begin{aligned} X^{a+1} 1 &= 0 \\ X^{a-1}\left(Y - \frac{\sum_{j=0}^{a-2}\eta^{j-m}}{\sum_{j=0}^{a-1}\eta^j} X\right) &= 0 \end{aligned} \tag{6.34}$$

and therefore, as an R-module

$$M \cong R/X^{a-1}R \oplus R/X^{a+1}R\,. \tag{6.35}$$

Thus, applying the defect corresponding to the matrix factorisation $P_{\{m,m+1\}}$ to the boundary condition associated to T_a $(0 < a < d)$ results in the boundary condition described by the matrix factorisation

$$P_{\{m,m+1\}} * T_a = T_{a-1} \oplus T_{a+1}\,. \tag{6.36}$$

If $a = 1$ or $a = d-1$ one of the summands is a trivial matrix factorisation. Using (6.28), one can obtain the action of arbitrary P_I to be

$$P_{\{m,\ldots,m+l\}} * T_a = \bigoplus_b \mathcal{N}_{la}^b T_b\,. \tag{6.37}$$

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6.1.3 Action of tensor product type defects

In this subsection we will discuss the action of tensor product (TP) type defects on boundary conditions and other TP type defects. Let us start with the discussion of the application of the defect corresponding to $T_{a,b}$ on the boundary condition represented by T_β. Indeed,

$$T_{a,b} * T_\beta = \big(T_a(X) \otimes \bar{T}_b(Y)\big) \otimes T_\beta(Y)\,. \tag{6.38}$$

Now, let us assume that the minimum $m = \min(b, \beta)$ satisfies $m \leq d - m$. This can always be achieved by shifting both $T_b \mapsto T_b[1]$ and $T_\beta \mapsto T_\beta[1]$, which does not affect (6.38). Consider the case $\beta = m$. The matrix factorisation (6.38) is isomorphic (up to trivial matrix factorisations) to the one arising from the resolution of the module

$$\begin{aligned} M &= \mathrm{coker}(t_1^{a,b} \otimes \mathrm{id}, -\mathrm{id} \otimes t_1^\beta) \\ &= \mathrm{coker}\begin{pmatrix} x^a & y^b & -y^\beta & 0 \\ y^{d-b} & x^{d-a} & 0 & -y^\beta \end{pmatrix} \\ &= \mathrm{coker}\begin{pmatrix} x^a & 0 & -y^\beta & 0 \\ 0 & x^{d-a} & 0 & -y^\beta \end{pmatrix} \\ &\cong \mathbb{C}[X,Y]/(X^a, Y^\beta) \oplus \mathbb{C}[X,Y]/(X^{d-a}, Y^\beta)\,, \end{aligned} \tag{6.39}$$

where it was used that $\beta \leq b, d-b$. Therefore for $\beta < d - \beta, b, d - b$

$$T_{a,b} * T_\beta = \Big(T^a \oplus T^{d-a}\Big)^{\oplus\beta} = (T^a \oplus T^a[1])^{\oplus\beta}\,. \tag{6.40}$$

If $b < \beta$ one can use the associativity of the tensor product of matrix factorisations to obtain a module M with cokernel representation as in (6.39) with β and b interchanged. (Also some irrelevant signs are different because one of the T_b, T_β appearing in (6.38) has a bar.) Thus, in the same way, one arrives at the result for arbitrary b and β:

$$T_{a,b} * T_\beta = (T^a \oplus T^a[1])^{\oplus \min(b,\beta,d-b,d-\beta)}\,. \tag{6.41}$$

Analogously one can deal with the composition of TP like defects to obtain

$$T_{a,b} * T_{\beta,\gamma} = (T_{a,\gamma} \oplus T_{a,\gamma}[1])^{\oplus \min(b,\beta,d-b,d-\beta)}\,. \tag{6.42}$$

Since $T_{\beta,\gamma} = T_\beta(Y) \otimes \bar{T}_\gamma(Z)$ is a tensor product matrix factorisation, this result indeed can be easily obtained from the action of $T_{a,b}$ on boundary conditions, namely

$$T_{a,b} * T_{\beta,\gamma} = (T_{a,b} * T_\beta) \otimes \bar{T}_\gamma\,, \tag{6.43}$$

and this trick can in fact also be used to deduce the action of permutation type defects on TP like defects from their action on boundary conditions:

$$P_I * T_{\beta,\gamma} = (P_I * T_\beta) \otimes \bar{T}_\gamma\,. \tag{6.44}$$

6.1.4 $W = X^d + Z^2$

Here we would like to extend the previous analysis to Landau-Ginzburg models with superpotentials $W = X^d + Z^2$. Defects between these models correspond to matrix factorisations of $X^d + Z^2 - Y^d - U^2$. The obvious generalisations of the factorisations (6.2) are just tensor products of the factorisations $P_I^d(X, Y)$ of $X^d - Y^d$ and factorisations $P_J^2(Z, U)$ of $Z^2 - U^2$. Obviously J can be chosen to consist either of 0 or 1, and these factorisations are symmetry defects with respect to the $\mathbb{Z}_2$ generated by changing the sign of the respective superfield. We denote the tensor products as

$$P_I^\pm := P_I^d(X, Y) \otimes P_{\{\pm 1 - 1\}}^2(Z, U) . \tag{6.45}$$

Note that not all of these factorisations are independent. Since the tensor product of two shifted matrix factorisations is equivalent to the tensor product of the unshifted ones, $P[1] \otimes Q[1] \cong P \otimes Q$, we have

$$P_I^\pm \cong P_{\{0,\ldots d-1\}-I}^\mp , \tag{6.46}$$

and all these defects can be expressed in terms of P_I^+ only. This is expected from Knörrer periodicity, which states that the category of matrix factorisations of a polynomial and that one of the same polynomial to which two squares are added are equivalent. In particular, the structure of defects in theories with superpotential $W = X^d$ and $W = X^d + Z^2$ should coincide.[15]

Because of the tensor product structure, composition of these defects can easily be reduced to the one of the tensor factors. Thus from (6.28) and the obvious $\mathbb{Z}_2$-composition of the symmetry defects, one deduces

$$P_{\{m_1,\ldots,m_1+l_1\}}^\sigma * P_{\{m_2,\ldots,m_2+l_2\}}^\rho = \bigoplus_l \mathcal{N}_{l_1 l_2}^l P_{\{\frac{1}{2}(l_1+l_2-l)+m_1+m_2,\ldots,\frac{1}{2}(l_1+l_2+l)+m_1+m_2\}}^{\sigma\rho} . \tag{6.47}$$

We would like to study how defects corresponding to these matrix factorisations act on boundary conditions. Corresponding to matrix factorisations of $X^d + Z^2$, the latter come in two classes [15]. Firstly, there are the obvious tensor product matrix factorisations

$$\Theta_a := T_a^d(X) \otimes T_1^2(Z) , \tag{6.48}$$

where as before

$$T_a^d(X) : \quad \mathbb{C}[X] \underset{X^{d-a}}{\overset{X^a}{\rightleftarrows}} \mathbb{C}[X] . \tag{6.49}$$

These factorisations are not "oriented" in the sense that $\Theta_a \cong \Theta_a[1] \cong \Theta_{d-a}$. The action of the defects $P_I^\pm$ on them can again be decomposed into the action of the respective tensor factors and with (6.37) one obtains

$$P_{\{m,\ldots,m+l\}}^\pm * \Theta_a = \bigoplus_b \mathcal{N}_{la}^b \Theta_b . \tag{6.50}$$

[15] Of course one can also study defects between Landau-Ginzburg models with superpotentials with $W = X^d$ and $W = X^d + Z^2$, which would then correspond to matrix factorisations of $X^d + Z^2 - Y^d$. The structure of these kinds of defects is different from the ones between models of the same type, but we will refrain from discussing them here.

For even d however the factorisations $\Theta_{\frac{d}{2}}$ are reducible. They split up

$$\Theta_{\frac{d}{2}} \cong \Psi^+ \oplus \Psi^- \tag{6.51}$$

into the two additional rank-one factorisations

$$\Psi^\pm : \quad \mathbb{C}[X,Z] \underset{\psi_0^\pm}{\overset{\psi_1^\pm}{\rightleftarrows}} \mathbb{C}[X,Z], \qquad \psi_1^\pm = \left(X^{\frac{d}{2}} \mp iZ\right), \quad \psi_0^\pm = \left(X^{\frac{d}{2}} \pm iZ\right). \tag{6.52}$$

In contrast to the Θ_i, the $\Psi^\pm$ are oriented; they satisfy $\Psi^\pm[1] \cong \Psi^\mp \ncong \Psi^\pm$, and the action of the defects associated to the $P_I^\pm$ on the corresponding boundary conditions is more complicated. $P_{\{m\}}^\pm$ for instance is a symmetry defect and as discussed in section 5 acts on any matrix factorisation by replacing

$$X \mapsto \eta^{-m} X, \quad Z \mapsto \pm Z. \tag{6.53}$$

In particular,

$$P_{\{m\}}^\sigma * \Psi^\rho = \Psi^{\sigma\rho\eta^{\frac{md}{2}}} = \begin{cases} \Psi^{\sigma\rho}, & m \text{ even} \\ \Psi^{-\sigma\rho}, & m \text{ odd} \end{cases}. \tag{6.54}$$

In view of the fact that also the $P_I^\pm$ with $|I| > 2$ are generated by the composition of those with $|I| \leq 2$ (c.f. (6.47)), we again only have to analyse the action of the $P_{\{m,m+1\}}^\pm$ on $\Psi^\pm$ by hand. To do this, we note that the result of $P_{\{m,m+1\}}^\sigma * \Psi^\rho$ is the matrix factorisation obtained from $R = \mathbb{C}[X,Z]/(X^d + Z^2)$-free resolutions of the module

$$\begin{aligned} M &= \mathbb{C}[X,Y,Z,U] / \left((Y^{\frac{d}{2}} - i\rho U),\ (Z - \sigma U),\ (X - \eta^m Y)(X - \eta^{m+1} Y) \right) && (6.55) \\ &\cong \mathbb{C}[X,Y,Z] / \left((Y^{\frac{d}{2}} - i\sigma\rho Z),\ (X - \eta^m Y)(X - \eta^{m+1} Y) \right). && (6.56) \end{aligned}$$

Because of the quadratic relation $(X - \eta^m Y)(X - \eta^{m+1} Y) = 0$ in M, ith powers of Y with $i \geq 2$ can be expressed as

$$Y^i = P_i(X) + Y Q_i(X). \tag{6.57}$$

Inductively one easily finds that

$$\begin{aligned} P_i(X) &= p_i X^i, \qquad p_i = -\eta^{-(m+1)i+1}(1 + \eta + \ldots + \eta^{i-2}), && (6.58) \\ Q_i(X) &= q_i X^{i-1}, \quad q_i = \eta^{-(m+1)(i-1)}(1 + \eta + \ldots + \eta^{i-1}). \end{aligned}$$

Therefore, M collapses to a submodule of $\mathbb{C}[X,Z] \oplus Y\mathbb{C}[X,Z]$, and the only task is to find the relations in it. These come from the relations

$$\begin{aligned} Y^{\frac{d}{2}-2} X^2 - \eta^m (1 + \eta) Y^{\frac{d}{2}-1} X + i\eta^{2m+1} \sigma\rho Z &= 0, && (6.59) \\ Y^{\frac{d}{2}-1} X^2 - i\eta^m (1 + \eta) \sigma\rho X Z + i\eta^{2m+1} \sigma\rho Y Z &= 0, && (6.60) \end{aligned}$$

which are obtained by substituting $Y^{\frac{d}{2}} = i\sigma\rho Z$ into $Y^i (X - \eta^m Y)(X - \eta^m Y) = 0$ for $i = \frac{d}{2} - 2$ and $i = \frac{d}{2} - 1$ respectively. Using (6.57), (6.58) and the explicit form of the p_i and q_i these equations can be written as

$$\begin{aligned} \left(q_{\frac{d}{2}-1} X^{\frac{d}{2}} + i\eta^{2m+1} \sigma\rho Z\right) + Y\left(-\eta^{2m+1} q_{\frac{d}{2}} X^{\frac{d}{2}-1}\right) &= 0, && (6.61) \\ \left(p_{\frac{d}{2}-1} X^{\frac{d}{2}+1} - i\eta^m (1 + \eta) \sigma\rho X Z\right) + Y\left(q_{\frac{d}{2}-1} X^{\frac{d}{2}} + i\eta^{2m+1} \sigma\rho Z\right) &= 0. && (6.62) \end{aligned}$$

Regarding $\mathbb{C}[X,Z]\oplus Y\mathbb{C}[X,Z]$ as $\mathbb{C}[X,Z]^2$, M is isomorphic to the cokernel of the matrix

$$O=\begin{pmatrix} q_{\frac{d}{2}-1}X^{\frac{d}{2}}+i\eta^{2m+1}\sigma\rho Z & p_{\frac{d}{2}-1}X^{\frac{d}{2}+1}-i\eta^{m}(1+\eta)\sigma\rho XZ \\ -\eta^{2m+1}q_{\frac{d}{2}}X^{\frac{d}{2}-1} & q_{\frac{d}{2}-1}X^{\frac{d}{2}}+i\eta^{2m+1}\sigma\rho Z \end{pmatrix}. \tag{6.63}$$

By means of elementary row and column transformations this matrix can be brought into the form

$$\begin{pmatrix} i\sigma\rho Z & X^{\frac{d}{2}+1}\left(p_{\frac{d}{2}-1}+\frac{q^2_{\frac{d}{2}-1}}{\eta^{2m+1}q_{\frac{d}{2}}}\right)+i\sigma\rho XZ\left(-\eta^m(1+\eta)+2\frac{q_{\frac{d}{2}-1}}{q_{\frac{d}{2}}}\right) \\ -q_{\frac{d}{2}}X^{\frac{d}{2}-1} & i\eta^{2m+1}\sigma\rho Z \end{pmatrix} \tag{6.64}$$

Using the explicit formulas for the q_i and p_i, in particular $q_{\frac{d}{2}}^{-1}=\frac{1}{2}(1-\eta)\eta^{(m+1)(\frac{d}{2}-1)}$, one can show that the upper right entry of this matrix indeed simplifies to

$$\frac{-\eta^{2m+1}}{q_{\frac{d}{2}}}X^{\frac{d}{2}+1}, \tag{6.65}$$

and again using elementary row and column transformations O can be brought into the form

$$O\mapsto\begin{pmatrix} X^{\frac{d}{2}-1} & -Z \\ Z & X^{\frac{d}{2}+1} \end{pmatrix}, \tag{6.66}$$

which is easily recognised as the matrix $\theta^1_{\frac{d}{2}-1}$ of the matrix factorisation $\Theta_{\frac{d}{2}-1}$. Thus,

$$M\cong\operatorname{coker}(\theta^1_{\frac{d}{2}-1}), \tag{6.67}$$

and

$$P^{\sigma}_{\{m,m+1\}}*\Psi^{\rho}=\Theta_{\frac{d}{2}-1}=\frac{1}{2}\sum_l\mathcal{N}_{1\frac{d}{2}}^{l}\Theta_l\,. \tag{6.68}$$

By means of the composition (6.47) this determines the action of all $P_I^{\pm}$ on the $\Psi^{\pm}$. For l_1 odd, it is straightforward to derive

$$P^{\sigma}_{\{m,\dots,m+l_1\}}*\Psi^{\rho}=\frac{1}{2}\sum_l\mathcal{N}_{l_1\frac{d}{2}}^{l}\Theta_l\,. \tag{6.69}$$

The simplest case for l_1 even is obviously $l_1=0$, which has been treated above, c.f. (6.54). The next simple case is $l_1=2$, for which the action of the defect can be obtained from

$$P^{\sigma}_{\{m,m+1\}}*P^{\sigma'}_{\{m',m'+1\}}\Psi^{\rho}=P^{\sigma}_{\{m,m+1\}}*\Theta_{\frac{d}{2}-1}=\Psi^+\oplus\Psi^-\oplus\Theta_{\frac{d}{2}-2}\,. \tag{6.70}$$

Here, we used that the factorisation $\Theta_{\frac{d}{2}}$ is reducible and can be decomposed into Ψ^+ and Ψ^-. Applying (6.47) we obtain

$$P^{\sigma\sigma'}_{\{m+m',m+m'+1,m+m'+2\}}*\Psi^{\rho}=\Psi^{(-1)^{m+m'}\sigma\sigma'\rho}\oplus\Theta_{\frac{d}{2}-2} \tag{6.71}$$

This immediately generalises to

$$P^{\sigma}_{\{m,\dots,m+l_1\}}*\Psi^{\rho}=\Psi^{(-1)^m\sigma\rho}\oplus\frac{1}{2}\sum_l\mathcal{N}_{\frac{d}{2}l_1}^{l}\Theta_l\quad\text{for } l_1 \text{ even}\,. \tag{6.72}$$

6.2 CFT approach

In the IR, the Landau-Ginzburg model with one chiral superfield and superpotential $W(X) = X^d$ and the one with an additional superfield and superpotential $W(X,Z) = X^d + Z^2$ both flow to versions of the unitary superconformal minimal model $\mathcal{M}_k$, $k = d-2$ with A-type modular invariant partition function. The two versions only differ in the definition of $(-1)^F$ on the Ramond-sectors.

The conformal field theories $\mathcal{M}_k$ are rational with respect to the $N=2$ super Virasoro algebra at central charge $c_k = \frac{3k}{k+2}$. In fact, the bosonic part of this algebra can be realised as the coset W-algebra

$$(\mathrm{SVir}_{c_k})_{\mathrm{bos}} = \frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{u}}(1)_4}{\widehat{\mathfrak{u}}(1)_{2k+4}}\,, \tag{6.73}$$

and the respective coset CFT can be obtained from $\mathcal{M}_k$ by a non-chiral GSO projection.

The Hilbert space $\mathcal{H}^k$ of $\mathcal{M}_k$ decomposes into irreducible highest weight representations of holomorphic and antiholomorphic super Virasoro algebras, but it is convenient to decompose it further into irreducible highest weight representations $\mathcal{V}_{[l,m,s]}$ of the bosonic subalgebra (6.73). These representations are labelled by

$$[l,m,s] \in \mathcal{I}_k := \{(l,m,s)\,|\, 0 \le l \le k,\, m \in \mathbb{Z}_{2k+4},\, s \in \mathbb{Z}_4,\, l+m+s \in 2\mathbb{Z}\}/\sim\,, \tag{6.74}$$

where $[l,m,s] \sim [k-l, m+k+2, s+2]$ is the field identification. The highest weight representations of the full super Virasoro algebra are given by

$$\mathcal{V}_{[l,m]} := \mathcal{V}_{[l,m,(l+m)\,\mathrm{mod}\,2]} \oplus \mathcal{V}_{[l,m,(l+m)\,\mathrm{mod}\,2+2]}\,. \tag{6.75}$$

For $(l+m)$ even $\mathcal{V}_{[l,m]}$ is in the NS-, for $(l+m)$ odd in the R-sector. Here $[l,m] \in \mathcal{J}_k := \{(l,m)\,|\,0 \le l \le k,\, m \in \mathbb{Z}_{2k+4}\}/\sim$, $[l,m] \sim [k-l, m+k+2]$. The Hilbert spaces of $\mathcal{M}_k$ in the NSNS- and RR-sectors then read

$$\mathcal{H}^k_{\mathrm{NSNS}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\mathrm{even}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,, \qquad \mathcal{H}^k_{\mathrm{RR}} \cong \bigoplus_{\substack{[l,m]\in\mathcal{J}_k \\ l+m\,\mathrm{odd}}} \mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}\,. \tag{6.76}$$

In this section we would like to discuss topological defects in the supersymmetric model $\mathcal{M}_k$ preserving B-type supersymmetry. Located on the real line $z = z^*$ they impose the following gluing conditions

$$\left.\begin{array}{r} T(z) - T(z^*) \\ \overline{T}(\bar z) - \overline{T}(\bar z^*) \\ G^{\pm}(z) - \eta G^{\pm}(z^*) \\ \overline{G}^{\pm}(\bar z) - \bar\eta \overline{G}^{\pm}(\bar z^*) \end{array}\right\} \to 0 \quad \text{for} \quad z - z^* \to 0\,, \tag{6.77}$$

for $\eta, \bar\eta \in \{\pm 1\}$. Representing the defects as operators $\mathcal{D}: \mathcal{H}^k \to \mathcal{H}^k$ the gluing conditions lead to commutation relations

$$\begin{gathered}[L_n, \mathcal{D}] = 0 = \left[\overline{L}_n, \mathcal{D}\right] \\ G^{\pm}_r \mathcal{D} - \eta \mathcal{D} G^{\pm}_r = 0 = \overline{G}^{\pm}_r \mathcal{D} - \bar\eta \mathcal{D} \overline{G}^{\pm}_r\,, \end{gathered} \tag{6.78}$$

for all $n \in \mathbb{Z}$ and all $r \in \mathbb{Z}+\frac{1}{2}$ ($r \in \mathbb{Z}$) in the NS- (R-) sectors. We will furthermore require $\mathcal{D}$ to commute with $(-1)^F$, which in general might be defined differently on both sides of the defect. In this paper, the discussion will be restricted to the case in which the action of $(-1)^F$ on both sides of the defect is the same, i.e. we will only discuss defects between the same type of models. Composing $\mathcal{D}$ with $(-1)^F$ results in an operator satisfying gluing conditions with opposite η and $\bar{\eta}$. Likewise, η and $\bar{\eta}$ can be changed separately if $(-1)^{F_L}$ and $(-1)^{F_R}$ are on their own symmetries of the theory.[16]

Since $\mathcal{M}_k$ is a diagonal RCFT with respect to the $N=2$ algebra, standard techniques can be used to construct the defect operators. First, Schur's lemma implies that for $\eta=\bar{\eta}$ $\mathcal{D}$ has to be a linear combination

$$\mathcal{D} = \sum_{[l,m]} \mathcal{D}^{[l,m]} \mathrm{P}_{[l,m]} = \mathcal{D}_{\mathrm{NSNS}} + \mathcal{D}_{\mathrm{RR}} \,. \tag{6.79}$$

of projectors $\mathrm{P}_{[l,m]}$ on the irreducible representations $\mathcal{V}_{[l,m]} \otimes \overline{\mathcal{V}}_{[l,m]}$. From this it is simple to obtain defects corresponding to other choices of η by composing with $(-1)^{F_L}$ or $(-1)^{F_R}$. Note that this formula combines the action of the defect on both NSNS- ($l+m$ even) as well as RR-sectors ($l+m$ odd). At this point we assume that we are dealing with defects between the same type of model, i.e. with the same definition of $(-1)^F$. Namely, in contrast to defects between the same version of minimal models, defects between the two different versions are linear combinations of intertwiners between representations $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,\bar{s}]}$ and $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,-\bar{s}]}$.

Indeed, also for the case of defects between the same version of minimal models it is useful to write the defect operators as sums over projectors $\mathrm{P}_{[l,m,s,\bar{s}]}$ of the modules $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,\bar{s}]}$ of the bosonic subalgebra

$$\mathcal{D} = \sum_{\substack{[l,m,s],\bar{s} \\ s-\bar{s}\,\mathrm{even}}} \mathcal{D}^{[l,m,s,\bar{s}]} \mathrm{P}_{[l,m,s,\bar{s}]} \,, \tag{6.80}$$

where it is understood that

$$\mathcal{D}^{[l,m,s+2,\bar{s}]} = \eta \mathcal{D}^{[l,m,s,\bar{s}]} \quad \text{and} \quad \mathcal{D}^{[l,m,s,\bar{s}+2]} = \bar{\eta} \mathcal{D}^{[l,m,s,\bar{s}]} \,. \tag{6.81}$$

The possible linear combinations of projectors are restricted by sewing relations which ensure that correlation functions do not depend on the different ways in which surfaces can be sewn together. In particular there is a sewing relation similar to Cardy's constraint for boundary conditions (see e.g. [1]). The standard solution, which can also be obtained via the folding trick from permutation boundary conditions is given by

$$\mathcal{D}^{[l,m,s,\bar{s}]}_{[L,M,S,\bar{S}]} = e^{-i\pi \frac{\bar{S}(s+\bar{s})}{2}} \frac{S_{[L,M,S-\bar{S}][l,m,s]}}{S_{[0,0,0],[l,m,s]}} \,, \tag{6.82}$$

where the different defects have been labelled by $[L,M,S,\bar{S}]$ with $[L,M,S-\bar{S}] \in \mathcal{I}_k$, and

$$S_{[L,M,S][l,m,s]} = \frac{1}{k+2} e^{-i\pi \frac{Ss}{2}} e^{i\pi \frac{Mm}{k+2}} \sin\left(\pi \frac{(L+1)(l+1)}{k+2}\right) \tag{6.83}$$

[16] In fact, the operators $(-1)^F$, $(-1)^{F_L}$ and $(-1)^{F_R}$ are indeed associated to topological defects as well, and composition with $\mathcal{D}$ can be interpreted as fusion of the respective defects.

is the modular S-matrix for the coset representations $\mathcal{V}_{[l,m,s]}$. Obviously, the possible choices of S and $\bar{S}$ are determined by η and $\bar{\eta}$ in the usual way, $\eta = (-1)^S$ and $\bar{\eta} = (-1)^{\bar{S}}$. The defect does not change under $(S, \bar{S}) \mapsto (S+2, \bar{S}+2)$.

Since these defects are topological we can bring them together to obtain new defects. From (6.78) it is clear that this operation preserves the gluing conditions so that the result will again be a B-type defect. The twist parameters η and $\bar{\eta}$ are multiplicative. On the level of defect operators this operation just amounts to their composition. Using the fact that the quantum dimensions (6.82) form representations of the respective fusion rules $\mathcal{N}$, one easily obtains the composition law

$$\begin{aligned}\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\mathcal{D}_{[L_2,M_2,S_2,\bar{S}_2]} &= \sum_{[L,M,S-\bar{S}]\in\mathcal{I}_k,\bar{S}} \mathcal{N}_{[L_1,M_1,S_1-\bar{S}_1][L_2,M_2,S_2-\bar{S}_2]}^{[L,M,S-\bar{S}]}\delta^{(4)}_{\bar{S}_1+\bar{S}_2,\bar{S}}\mathcal{D}_{[L,M,S,\bar{S}]} \\ &= \sum_L \mathcal{N}_{L_1 L_2}^{L}\mathcal{D}_{[L,M_1+M_2,S_1+S_2,\bar{S}_1+\bar{S}_2]}\,. \end{aligned} \tag{6.84}$$

Note that for $L=0$ these defects are group-like. The defect labels $[0, M, S, \bar{S}]$ correspond to simple currents, and their fusion determines the composition of the corresponding defects.

6.2.1 Action on boundary conditions

Next, we would like to discuss the action of these topological B-type defects on B-type boundary conditions. On the real line the latter impose gluing conditions

$$\left.\begin{aligned} T(z) - \overline{T}(\bar{z}) \\ G^{\pm}(z) - \eta\overline{G}^{\pm}(\bar{z}) \end{aligned}\right\} \to 0 \quad \text{for} \quad z - z^* \to 0\,, \tag{6.85}$$

translating into the relations

$$\begin{aligned} (L_n - \overline{L}_{-n})\|B\rangle\!\rangle &= 0\,. \\ (G_r^{\pm} - i\eta\overline{G}^{\pm}_{-r})\|B\rangle\!\rangle &= 0 \end{aligned} \tag{6.86}$$

for the respective boundary states $\|B\rangle\!\rangle$. The choice of sign $\eta \in \{\pm 1\}$ in the gluing conditions for the supercurrents corresponds to the choice of different spin structures. Modules $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,s]}$ support Ishibashi states $|[l,m,s]\rangle\!\rangle_B$ solving the gluing conditions (6.86) if $[l,m,s] \sim [l,-m,-\bar{s}]$. Thus, there are Ishibashi states $|[l,0,s]\rangle\!\rangle_B$ for all $[l,0,s] \in \mathcal{I}_k$. In case k is even there are additional Ishibashi states $|[\frac{k}{2}, \frac{k+2}{2}, 1]\rangle\!\rangle_B$.

Apart from the gluing conditions (6.86) above, the boundary condition should also preserve $\mathbb{Z}_2$-fermion number, which means that $(-1)^F\|B\rangle\!\rangle = \|B\rangle\!\rangle$. Since the two CFTs corresponding to the Landau-Ginzburg models with superpotentials $W = X^d$ and $W = X^d + Z^2$ differ by the definition of $(-1)^F$, we have to treat the two cases separately.

Case 1: $W = X^d$. Let us start with the CFT associated to the superpotential $W = X^d$. In this model $(-1)^F$ acts on $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,\bar{s}]}$ as multiplication by $(-1)^{\frac{s+\bar{s}}{2}}$, and hence only Ishibashi states $|[l,0,s]\rangle\!\rangle_B$ can contribute to B-type boundary states.[17] The standard

[17] The relevant GSO-projection in this model is of type $0A$ projecting onto the subspace $\mathcal{H}_k^{0A} \cong \bigoplus(\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,-s]})$.

construction yields boundary states

$$\|[L,M,S]\rangle\!\rangle_B^{\rm NS} = \|[L,M+2,S]\rangle\!\rangle_B^{\rm NS} = \sqrt{2(k+2)} \sum_{\substack{[l,0,s]\in\mathcal{I}_k \\ s \text{ even}}} \frac{S_{[L,M,S][l,0,s]}}{\sqrt{S_{[0,0,0][l,0,s]}}} |[l,0,s]\rangle\!\rangle_B \tag{6.87}$$

$$\|[L,M,S]\rangle\!\rangle_B^{\rm R} = \|[L,M+2,S]\rangle\!\rangle_B^{\rm R} = \sqrt{2(k+2)} \sum_{\substack{[l,0,s]\in\mathcal{I}_k \\ s \text{ odd}}} \frac{S_{[L,M,S][l,0,s]}}{\sqrt{S_{[0,0,0][l,0,s]}}} |[l,0,s]\rangle\!\rangle_B \,,$$

for every $[L,M,S] \in \mathcal{I}_k$, where we have specified both, the NSNS- as well as the RR-components. The boundary states in the GSO projected theory can be obtained by adding RR- and NSNS-part of the boundary state with a normalisation factor $\frac{1}{\sqrt{2}}$.

A shift by 2 in the S labels inverts the sign in front of the RR-sector Ishibashi states and hence corresponds to a brane-anti-brane map. Similarly as in the defect case, S mod 2 is given by $\eta = (-1)^S$ in the gluing conditions (6.85) above.

In case k is odd, all boundary states are oriented, i.e. they have non-trivial RR-components and are therefore not invariant under the brane-anti-brane map. If k is even, the boundary states $\|[\frac{k}{2}, \frac{k}{2} - S, S]\rangle\!\rangle_B$ have vanishing RR-component, and are therefore unoriented.

Moving the topological B-type defects constructed above to a boundary with B-type boundary condition amounts to applying the corresponding defect operators $\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}$ to the respective boundary state $\|[L_2,M_2,S_2]\rangle\!\rangle_B$. From (6.78) and (6.86) it is obvious that the resulting states again satisfy B-type gluing conditions and preserve $(-1)^F$. Furthermore, sewing relations ensure that these states are again boundary states. Direct calculation yields

$$\begin{aligned}\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[L_2,M_2,S_2]\rangle\!\rangle_B &= \sum_{[L,M,S]\in\mathcal{I}_k} \mathcal{N}_{[L_1,M_1,S_1-\bar{S}_1][L_2,M_2,S_2]}^{[L,M,S]} \|[L,M,S]\rangle\!\rangle_B \\ &= \sum_L \mathcal{N}_{L_1L_2}^{L} \|[L, M_1+M_2, S_1-\bar{S}_1+S_2]\rangle\!\rangle_B \,.\end{aligned} \tag{6.88}$$

From this one immediately deduces that defects with $S_1 - \bar{S}_1 = 0$ map branes to branes, hence are orientation preserving, whereas defects with $S_1 - \bar{S}_1 = 2$ reverse brane orientation. Defects with odd $S_1 - \bar{S}_1$ flip the spin structure compatible with the boundary condition.

Case 2: $W = X^d + Z^2$. In the model corresponding to $W = X^d + Z^2$, $(-1)^F$ acts on $\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,\bar{s}]}$ as multiplication by $(-1)^{\frac{s-\bar{s}}{2}}$, which only leaves the Ishibashi states $|[l,0,s]\rangle\!\rangle_B$ for even s and $|[\frac{k}{2}, \frac{k+2}{2}, \pm 1]\rangle\!\rangle_B$ invariant.[18] Since the models corresponding to the superpotentials $W = X^d$ and $W = X^d + Z^2$ are $\mathbb{Z}_2$-orbifolds of each other [15], the boundary states of one of the models can be obtained from the ones of the other by means of a standard orbifold construction. Applying this construction to the boundary states (6.87)

[18] The corresponding GSO-projection is of type $0B$ and projects onto the subspace $\mathcal{H}_k^{0B} \cong \bigoplus(\mathcal{V}_{[l,m,s]} \otimes \overline{\mathcal{V}}_{[l,m,s]})$.

one obtains [33]

$$
\begin{aligned}
\|[L,M,S]\rangle\!\rangle_B^{\rm NS} &= \|[L,M+2,S]\rangle\!\rangle_B^{\rm NS} = \|[L,M,S+2]\rangle\!\rangle_B^{\rm NS} \\
&= 2^{1-\delta_{L,\frac{k}{2}}}\sqrt{k+2}\sum_{\substack{[l,0,s]\in\mathcal{I}_k\\ s\ \text{even}}}\frac{S_{[L,M,S][l,0,s]}}{\sqrt{S_{[0,0,0][l,0,s]}}}|[l,0,s]\rangle\!\rangle_B \\
\|[L,M,S]\rangle\!\rangle_B^{R} &= \|[L,M+2,S]\rangle\!\rangle_B^{R} = \delta_{L,\frac{k}{2}}\sqrt{2(k+2)}e^{-\frac{i\pi S^2}{2}}\sum_{s=\pm1}e^{-\frac{i\pi Ss}{2}}|[\frac{k}{2},\frac{k+2}{2},s]\rangle\!\rangle_B\,.
\end{aligned}
\tag{6.89}
$$

Note that for $L\neq\frac{k}{2}$, the $\mathbb{Z}_2$-orbifold projects out the RR-components of the respective boundary states, so that $\|[L,M,S]\rangle\!\rangle_B^R=0$ for $L\neq\frac{k}{2}$. Thus, the boundary states associated to such $[L,M,S]$ are not oriented, and only depend on S mod 2, which distinguishing the spin structures $\eta=(-1)^S$. Only in case of even k do there exist oriented boundary states. These emanate from boundary states with $L=\frac{k}{2}$ in the unorbifolded theory which are invariant under the orbifold group and therefore pick up twisted RR-sector contributions upon orbifolding. They are not invariant with respect to $S\mapsto S+2$.

Since the boundary states (6.89) with $L\neq\frac{k}{2}$ just correspond to $\mathbb{Z}_2$-orbits of boundary states (6.87) of the unorbifolded model, one can immediately conclude from (6.88) that the action of the defects on these states is given by

$$
\begin{aligned}
\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[L_2,M_2,S_2]\rangle\!\rangle_B &= \sum_{L\neq\frac{k}{2}}\mathcal{N}_{L_1L_2}^{L}\|L,M_1+M_2,S_1+\bar{S}_1+S_2\rangle\!\rangle_B \\
&\quad+N_{L_1L_2}^{\frac{k}{2}}\Big(\|[\frac{k}{2},M_1+M_2,S_1+\bar{S}_1+S_2]\rangle\!\rangle_B \\
&\qquad\qquad+\|[\frac{k}{2},M_1+M_2,S_1+\bar{S}_1+S_2+2]\rangle\!\rangle_B\Big)
\end{aligned}
\tag{6.90}
$$

No RR-sector contribution can arise, and therefore only unoriented boundary states can emerge from this operation. In particular, if $k/2$ is contained in the fusion of L_1 and L_2 the sum of the two short orbit boundary states appears. Since the branes remain unoriented, defects whose $S_1+\bar{S}_1$ differs by 2 act in the same way.

More interesting is the action on the unoriented boundary states with $L=\frac{k}{2}$. The action of a defect $\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}$ on the NS-component of a boundary state $\|[\frac{k}{2},M,S]\rangle\!\rangle_B$ is simply given by one half of (6.90) with $L_2=k/2$. Note that if L appears in the fusion of $k/2$ with L_1 so does $k-L$. This means that the action of the defect on the NS-component of the oriented boundary state with $L=\frac{k}{2}$ produces a sum with unit coefficients of unoriented boundary states with $L\neq k/2$. For odd L_1, this is already the full story, since $k/2$ does not appear in the fusion of $k/2$ with L_1. Furthermore, defects with odd L_1 annihilate the RR-component of the boundary state due to the $\widehat{\mathfrak{su}}(2)_k$-part of the S-matrix. Hence for L_1 odd

$$
\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[\frac{k}{2},M_2,S_2]\rangle\!\rangle_B=\frac{1}{2}\sum_{L}\mathcal{N}_{L_1\frac{k}{2}}^{L}\|[L,M_1+M_2,S_1+\bar{S}_1+S_2]\rangle\!\rangle_B\,. \tag{6.91}
$$

On the other hand, if L_1 is even, the fusion of L_1 with $k/2$ will again contain $k/2$, and instead of annihilating the RR-component of the boundary state, the defect operator multiplies it the respective Ishibashi states $|[\frac{k}{2},\frac{k+2}{2},s]\rangle\!\rangle_B$ by $(-1)^{\frac{L_1+M_1-(S_1+\bar{S}_1)s}{2}}$. (Recall L_1

and therefore $M_1 - S_1 - \bar{S}_1$ are even, and s is odd.) It is then clear that the defect will change the spin structure according to (S_2 mod 2) $\mapsto$ ($S_1 + \bar{S}_1 + S_2$ mod 2). Whether the resulting boundary state has S-label $S_1 + \bar{S}_1 + S_2$ or $S_1 + \bar{S}_1 + S_2 + 2$ is determined by the overall sign of the RR-component. Altogether, for L_1 even one arrives at

$$\begin{aligned}\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[\frac{k}{2}, M_2, S_2]\rangle\rangle_B &= \frac{1}{2}\sum_{L\neq k/2}\mathcal{N}_{L_1\frac{k}{2}}^{L}\|[L, M_1 + M_2, S_1 + \bar{S}_1 + S_2]\rangle\rangle_B \\ &+\|[\frac{k}{2}, M_1{+}M_2, S_1{+}\bar{S}_1{+}(-1)^{S_1+\bar{S}_1}S_2{-}L_1{-}M_1{-}(S_1{+}\bar{S}_1)^2]\rangle\rangle_B\,.\end{aligned} \tag{6.92}$$

The orientation of the oriented boundary state appearing in this composition depends on the defect labels and S_2 in a rather complicated way. In the case that the defect preserves the spin structure of the boundary state (that is $S_1 + \bar{S}_1$ is even, and hence also $L_1 + M_1$ even), the corresponding S-label is given by $S_1 + \bar{S}_1 + S_2 - L_1 - M_1$. If on the other hand the defect changes the spin-structure, i.e. $S_1 + \bar{S}_1$ is odd, the resulting S-label becomes $S_1 + \bar{S}_1 - S_2 - L_1 - M_1 - 1$.

A-type boundary states. et us close the discussion of the conformal field theory of topological defects by noting that since these defects are topological, they also act naturally on A-type boundary states. These satisfy gluing relations

$$\begin{aligned}(L_n - \overline{L}_{-n})\|A\rangle\rangle &= 0\,. \\ (G_r^{\pm} - i\eta\overline{G}_{-r}^{\mp})\|A\rangle\rangle &= 0\end{aligned} \tag{6.93}$$

and, in the theory corresponding to $W = X^d + Z^2$, are given by the standard Cardy boundary states

$$\begin{aligned}\|[L,M,S]\rangle\rangle_A^{\mathrm{NS}} &= \sqrt{2}\sum_{\substack{[l,m,s]\\ s\text{ even}}}\frac{S_{[L,M,S][l,m,s]}}{\sqrt{S_{[0,0,0][l,m,s]}}}|[l,m,s]\rangle\rangle_A\,, \\ \|[L,M,S]\rangle\rangle_A^{R} &= \sqrt{2}\sum_{\substack{[l,m,s]\\ s\text{ odd}}}\frac{S_{[L,M,S][l,m,s]}}{\sqrt{S_{[0,0,0][l,m,s]}}}|[l,m,s]\rangle\rangle_A\,.\end{aligned} \tag{6.94}$$

The discussion of the action of the topological defects on these boundary states is similar to the one for the B-type boundary states, with the result

$$\mathcal{D}_{[L_1,M_1,S_1,\bar{S}_1]}\|[L_2, M_2, S_2]\rangle\rangle_A = \sum_L \mathcal{N}_{L_1L_2}^{L}\|[L, M_1 + M_2, S_1 + \bar{S}_1 + S_2]\rangle\rangle_A\,. \tag{6.95}$$

We omit a discussion for the A-type boundary states in the theory with the other definition of the fermion number. Let us just mention that one can again use orbifold techniques to construct the boundary states from the ones given above. Since none of the states (6.94) is invariant under the respective orbifold group, they do not get twisted sector contributions in the orbifolding construction. Instead they can all be represented as orbits under the orbifold group of the boundary states in the unorbifolded theory. Hence, the action of the defects on these boundary conditions can be easily deduced from (6.95).

6.2.2 Comparison to the Landau-Ginzburg analysis

We can now compare the results to the Landau-Ginzburg analysis of section 6.1. Matrix factorisations of type P_I for I consisting of consecutive integers modulo d are known to correspond to permutation boundary conditions in the tensor product of minimal models [34, 23]. The folding trick therefore implies that the P_I indeed correspond to the topological defects constructed above. More precisely:

$$\begin{aligned} P_{\{m,m+1,\ldots,m+l\}} &\leftrightarrow \mathcal{D}_{[l,l+2m,0,0]}\,, \\ P^{\pm}_{\{m,m+1,\ldots,m+l\}} &\leftrightarrow \mathcal{D}_{[l,l+2m,1\mp1,0]}\,. \end{aligned} \tag{6.96}$$

The folding trick guarantees that this identification is compatible with the topological spectra. Comparing the formula (6.28) for compositions of the defects P_I in Landau-Ginzburg models with the formula (6.84) for the composition of the topological defects $\mathcal{D}_{[l,l+2m,S,0]}$, $S \in \{0,2\}$ in minimal models one indeed also finds agreement.

Using the correspondence [12]

$$T_l \leftrightarrow \|[l-1,l-1,0]\rangle\rangle_B \tag{6.97}$$

between matrix factorisations of $W = X^d$ and boundary conditions in minimal models, one easily observes that the agreement found for the composition of defects also holds for the action of defects on boundary conditions (c.f. equations (6.37) and (6.88)).

This extends to the defect action on the tensor product factorisations Θ_l, in models with superpotentials $W = X^d + Z^2$. As has been discussed in [15] they correspond to the unoriented "long orbit" boundary states with $L \neq \frac{k}{2}$ in (6.89):

$$\begin{aligned} \Theta_l &\leftrightarrow \|[l-1,l-1,0]\rangle\rangle_B \quad \text{for } l \neq \frac{d}{2}\,, \\ \Theta_{\frac{d}{2}} &\leftrightarrow \|[\frac{d}{2}-1,\frac{d}{2}-1,0]\rangle\rangle_B + \|[\frac{d}{2}-1,\frac{d}{2}-1,2]\rangle\rangle_B\,, \end{aligned} \tag{6.98}$$

and a comparison between (6.50) and (6.90) shows agreement for the defect action on these. The matrix factorisations $\Psi^{\pm}$ on the other hand which exist for even d correspond to the oriented "short orbit" boundary states with $L = \frac{k}{2}$ in (6.89) [15]

$$\Psi^{\pm} \leftrightarrow \|[\frac{d}{2}-1,\frac{d}{2}-1,1\mp1]\rangle\rangle_B\,. \tag{6.99}$$

Also for these boundary conditions the defect action derived in the Landau-Ginzburg framework (6.72), (6.69) agrees with the one found in the conformal field theory (6.92), (6.91).

Let us close this discussion by noting that the matrix factorisations $T_{i,j}$ are indeed tensor product matrix factorisations.[19] The latter are known to correspond to the respective tensor product boundary states in the IR. The folding trick therefore implies the identification of these matrix factorisations with the completely reflective conformal B-type defects

$$T_{i,j} \leftrightarrow \|[i-1,i-1,0]\rangle\rangle\langle\langle[j-1,j-1,0]\| \tag{6.100}$$

[19] We discuss these types of defects in the model corresponding to $W = X^d$. The discussion immediately carries over to the model associated to $W = X^d + Z^2$.

in the minimal models $\mathcal{M}_k$. That this identification is compatible with the topological spectra is clear from the folding trick. Since these defects are not topological their composition and action on boundary conditions is not well-defined in the CFT.

7. Discussion

In this paper, we have discussed B-type defects in the context of Landau-Ginzburg theories. Those defects between models with superpotential W_1 and W_2 can be described by matrix factorisations of $W_1 - W_2$. We have discussed how two such defects can merge, and how they act on B-type boundary conditions, which in turn have a description in terms of matrix factorisations of the individual superpotentials W_i. These two operations turn out to be quite similar, namely, they are both given by taking the tensor product of the matrix factorisations describing defects and boundary conditions respectively. The resulting factorisations are a priori infinite dimensional, but can be reduced to finite dimensional ones by splitting off infinitely many brane-anti-brane pairs. We have described a method of how to obtain the reduced factorisations without going through the explicit reduction procedure.

We have discussed the special defects arising from symmetries of the bulk theories, and compared in detail the description of B-type defects in Landau-Ginzburg models with superpotentials $W = X^d$, $W = X^d + Z^2$, with the one of defects in the corresponding IR CFTs.

As a next step it would now be interesting to extend the analysis to charge-projected Landau-Ginzburg models with several superfields, which in the IR flow to superconformal field theories with $c = 9$ and describe the stringy regime of Calabi-Yau compactifications. Of course, the matrix factorisations for Landau-Ginzburg models with more chiral superfields are more complicated, but at least for models where the superpotential is a Fermat polynomial the factorisations described here can be used as building blocks. Furthermore, the orbifold construction introduces more structure, because it makes it necessary to consider graded matrix factorisations [35, 36].

Since certain orbifolds of Landau-Ginzburg models have a geometric interpretation as sigma model with target space X, the projective variety defined by the vanishing of the superpotential, the question about the geometric realisation of the defects and the D-branes they act on arises. For D-branes, the connection between matrix factorisations and large volume geometry has been investigated in [37−40, 22, 41]. A first idea of a geometric realisation of the defects can be obtained via the folding trick, according to which defects connecting two sigma models with target spaces X, Y correspond to B-type D-branes on the product $X \times Y$. The respective D-brane category in the topologically twisted theory can be described by $D^\flat(Coh(X \times Y))$, the derived category of coherent sheaves on the product space.

According to our general discussion, we expect that the defects act on D-branes and hence should provide transformations from $D^\flat(Coh(X))$ to $D^\flat(Coh(Y))$, and indeed one can associate to any element $\Phi \in D^\flat(Coh(X \times Y))$ a Fourier-Mukai transformation with kernel Φ.[20] Conversely, it has been shown [43, 44] that any equivalence $D^\flat(Coh(X)) \to D^\flat(Coh(Y))$ can be written as a Fourier- Mukai transformation. It therefore seems plausible that defects have a natural interpretation as Fourier-Mukai transformations at large volume.

For some simple transformations this is indeed the case. For instance, in Landau-Ginzburg orbifolds, there is a quantum symmetry which is broken once one moves away from the Landau-Ginzburg point in the bulk moduli space. We can associate a symmetry defect to this operation, which acts on the D-branes in the Landau-Ginzburg model. This quantum symmetry is known to correspond to the B-brane monodromy transformation around the Landau-Ginzburg point, which in the geometric context can be realised by a Fourier-Mukai transformation. We hope to come back to this issue in the future.

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A. Spectra of symmetry defects

Let $W(X_i)$ be a polynomial in the variables $X_1, \ldots, X_n$. Furthermore let Γ be a group acting linearly and unitarily on the space spanned by the X_i. Then for any $g \in \Gamma$, the polynomial $W(X_i) - W(Y_i)$ can be written as in (5.8) leading to the matrix factorisation D_g of (5.9). Here we would like to outline the calculation of the BRST-cohomology $\mathcal{H}^*(D_g, D_1)$. For this let $Q := D_g$ and $R := \mathbb{C}[X_i, Y_i]/(W(X_i) - W(Y_i))$. As used in section 4.2,

$$\mathcal{H}^i(Q, D_1) \cong \mathrm{Ext}_R^{2+2n+i}(\mathrm{coker}(q_1), R/(X_i - Y_i)) \,. \tag{A.1}$$

The Ext-groups can be calculated as the cohomology of the sequence obtained by applying the functor $\mathrm{Hom}(\cdot, R/(X_i - Y_i))$ to the the R-free resolution of $\mathrm{coker}(q_1)$ given by the matrix factorisation Q. But this sequence can be written as

$$\ldots \xrightarrow{\widetilde{q}_1} (R/(X_i - Y_i))^{2^n} \xrightarrow{\widetilde{q}_0} (R/(X_i - Y_i))^{2^n} \xrightarrow{\widetilde{q}_1} (R/(X_i - Y_i))^{2^n} \longrightarrow 0 \,, \tag{A.2}$$

where $\widetilde{q}_a = q_a(X_j, Y_j = X_j)$. Let us assume that g acts diagonally on the X_i. Then, setting $Y_j = X_j$ in each of the tensor factors P^i of D_g amounts to $\widetilde{p}^i_1 = 0$, $\widetilde{p}^i_0 = A_i(X_j, Y_j = X_j)$ in case X_i is g-invariant, and $\widetilde{p}^i_1 = (1-g)(X_i)$, $\widetilde{p}^i_0 = 0$ otherwise.[21] From this it is obvious that $\ker(\widetilde{q}_a)$ is non-trivial only if $a + |\{j | X_j \neq g(X_j)\}|$ is even, in which case the kernel is just

$$\ker(\widetilde{q}_a) \cong R/(X_i - Y_i)R \tag{A.3}$$

[20] A Landau-Ginzburg realisation of certain Fourier-Mukai transformations , namely monodromy actions, has been discussed in [42].

[21] The latter can always be achieved by means of an equivalence transformation.

and

$$\begin{aligned}\mathrm{im}(\widetilde{q}_{a+1}) = &\sum_{X_j \neq g(X_j)} (1-g)(X_j)R/(X_i - Y_i)R \\ &+ \sum_{X_j = g(X_j)} A_j(X_i, Y_i = g(X_i))R/(X_i - Y_i)R\,. \end{aligned} \tag{A.4}$$

Thus for $a = |\{j|X_j \neq g(X_j)\}| =: N_{\mathrm{n-inv}}$

$$\ker(\widetilde{q}_a)/\mathrm{im}(\widetilde{q}_{a+1}) \cong \mathbb{C}[X_j^{\mathrm{inv}}]/(\partial_j W_{\mathrm{inv}})\,, \tag{A.5}$$

where X_j^{inv} are the g-invariant variables and W_{inv} is obtained from W by setting all non-invariant variables to zero. Therefore we obtain

$$\begin{aligned}\mathcal{H}^{N_{\mathrm{n-inv}}}(D_g, D_1) &\cong \mathbb{C}[X_j^{\mathrm{inv}}]/(\partial_j W_{\mathrm{inv}})\,, \\ \mathcal{H}^{N_{\mathrm{n-inv}}+1}(D_g, D_1) &\cong \{0\}\,. \end{aligned} \tag{A.6}$$

The result can be summarised as follows: every state in $\mathcal{H}^*(D_g, D_1)$ can be written as $p(X_i^{\mathrm{inv}}) \prod_{j:X_j \neq g(X_j)} \omega_j$, where ω_j are fermions associated to every non-g-invariant variable X_j, and $p \in \mathbb{C}[X_i^{\mathrm{inv}}]/(\partial_i W_{\mathrm{inv}})$ is a polynomial in the g-invariant variables $X_j^{\mathrm{inv}} = g(X_j^{\mathrm{inv}})$. W_{inv} is obtained from W by setting all non-invariant variables to zero. This is in agreement with the g-twisted bulk Hilbert spaces obtained in [25, 24].

B. Explicit equivalence for $D_1 \otimes T_1$

Here we would like to show explicitly that the infinite dimensional matrix factorisation

$$D_1 \otimes T_1(Y): \quad r_1 = \begin{pmatrix} X - Y & -Y \\ Y^{d-1} & \frac{X^d - Y^d}{X-Y} \end{pmatrix}, \quad r_0 = \begin{pmatrix} \frac{X^d - Y^d}{X-Y} & Y \\ -Y^{d-1} & X - Y \end{pmatrix} \tag{B.1}$$

of X^d over $\mathbb{C}[X]$, which is obtained as the tensor product of the matrix factorisations

$$D_1: \quad p_1 = (X - Y),\ p_0 = \frac{X^d - Y^d}{X - Y} \tag{B.2}$$

of $X^d - Y^d$ and

$$T_1(Y): \quad q_1 = Y,\ q_0 = Y^{d-1} \tag{B.3}$$

of Y^d is indeed equivalent to $T_1(X)$. Using the trick discussed in section 4.2 one easily arrives at this conclusion, because $D_1 \otimes T_1(Y)$ has to be equivalent to the matrix factorisation obtained from the $R = \mathbb{C}[X]/(X^d)$-free resolution of the module $M := \mathrm{coker}(X - Y, Y) \cong \mathrm{coker}(X, Y) \cong R/XR$ by chopping off an even number of terms. But obviously M has an R-free resolution given by $T_1(X)$.

To construct the equivalence explicitly note first that by means of

$$u_0 = \begin{pmatrix} 1 & 0 \\ \frac{1}{X}\left(\frac{X^d - Y^d}{X-Y} - Y^{d-1}\right) & 1 \end{pmatrix}, \quad v_0 = u_0^{-1}\,, \tag{B.4}$$

$$u_1 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \qquad v_1 = u_1^{-1} \tag{B.5}$$

(r_1, r_0) is equivalent to

$$r_1' = \begin{pmatrix} X & Y \\ 0 & -X^{d-1} \end{pmatrix}, \quad r_0' = \begin{pmatrix} X^{d-1} & Y \\ 0 & -X \end{pmatrix}. \tag{B.6}$$

Regarding $\mathbb{C}[X, Y]$ as the infinite dimensional free $\mathbb{C}[X]$-module $\mathbb{C}[X, Y] \cong \mathbb{C}[X] + Y\mathbb{C}[X] + Y^2\mathbb{C}[X] + \ldots$, Y can be represented by the infinite dimensional matrix

$$Y = \begin{pmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{pmatrix}. \tag{B.7}$$

Using this representation r_1' takes the form

$$r_1' = \left(\begin{array}{ccc|ccc} X & & & 0 & & \\ & \ddots & & 1 & \ddots & \\ & & \ddots & & \ddots & \ddots \\ \hline & & & -X^{d-1} & & \\ & & & & \ddots & \\ & & & & & \ddots \end{array}\right) \tag{B.8}$$

Now one easily finds the following chain of elementary row and column transformations of r_1':

$$\begin{aligned} r_1' &\mapsto \left(\begin{array}{ccc|ccc} X & & & 0 & & \\ & 1 & & X & \ddots & \\ & & \ddots & & \ddots & \ddots \\ \hline 0 & -X^{d-1} & & & & \\ & \ddots & \ddots & & & \\ & & \ddots & & & \end{array}\right) \mapsto \left(\begin{array}{ccc|ccc} X & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ \hline 0 & -X^{d-1} & & X^d & & \\ & \ddots & \ddots & & \ddots & \\ & & \ddots & & & \ddots \end{array}\right) \\ &\mapsto \left(\begin{array}{ccc|ccc} X & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ \hline & & & X^d & & \\ & & & & \ddots & \\ & & & & & \ddots \end{array}\right) \end{aligned} \tag{B.9}$$

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The opposite transformations lead to

$$r_0' \mapsto \left(\begin{array}{ccc|ccc} X^{d-1} & & & & & \\ & X^d & & & & \\ & & \ddots & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{array}\right), \tag{B.10}$$

and hence we have obtained an explicit equivalence of the infinite dimensional matrix factorisation $D_1 \otimes T_1(Y)$ to the sum of the matrix factorisation $T_1(X)$ with infinitely many trivial matrix factorisations $(1, X^d)$.

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